6.1 Revision

Here is the formal definition of a group;

The Definition of a Group

If G is a set and $*$ is a binary operation defined on G, then $(G, *)$ is a group if the following four axioms hold:

- **Closure:** For all $a, b \in G, a * b \in G$
- **Identity:** There exists an identity element $e \in G$.
  This is such that, for all $a \in G$, $a * e = e * a = a$
- **Inverses:** For each $a \in G$, there exists an inverse element $a^{-1} \in G$
  This is such that $a * a^{-1} = a^{-1} * a = e$
- **Associativity:** For all $a, b, c \in G$, $a * (b * c) = (a * b) * c$

And a reminder of Lagrange's theorem;

Lagrange's Theorem

If H is a subgroup of a finite group G, the order of H divides the order of G.

6.2 Exercise

Marks Available: 40

Question 1

Solve the following congruence equation modulo 8.
Any answers should be given as elements from the set of least residues modulo 8.

$$2x - 9 \equiv 5 \pmod 8$$

[3 marks]

Question 2

Consider the following binary operation table;

∘	*a*	*b*	*c*	*d*
a	*c*	*d*	*c*	*b*
b	*d*	*c*	*b*	*a*
c	*a*	*d*	*c*	*d*
d	*b*	*a*	*b*	*c*

(a) Explain why the set of elements $\{a, b, c, d\}$ do **not** form a group.

[1 mark]

(b) Given the binary operation table, calculate

(i) $c \circ d$

[1 mark]

(ii) $a \circ (c \circ b)$

[1 mark]

(iii) $(a \circ c) \circ b$

[1 mark]

(iv) $(d \circ c) \circ (b \circ a)$

[1 mark]

(c) **(i)** Does the expression $a \circ b \circ c$ make sense ?

State your answer as either "Yes" or "No"

[1 mark]

(ii) Give a reason for your (c)(i) answer.

[1 mark]

Question 3

The set $G = \{ 1, 3, 5, 9, 11, 13 \}$ forms a group under multiplication modulo 14.

(i) Complete the following Cayley table for this group;

$\times_{14}$	1	3	5	9	11	13
1			5			
3		9				
5						
9				11		
11		5				
13						1

[2 marks]

(ii) Explain why $(G, *)$ can have no subgroup of order 4

[2 marks]

(iii) Find a subgroup of order 2 and construct its Cayley table.

$\times_{14}$		

[1 mark]

(iv) Find a subgroup of order 3 and construct its Cayley table.

$\times_{14}$			

[2 marks]

Question 4

Further A-Level, Pure Mathematics 2, June 2022, Paper 4A, Q1 (Edexcel)

The group S_4 is the set of all possible permutations that can be performed on the four numbers 1, 2, 3 and 4, under the operation of composition.

For the group S_4

(a) write down the identity element,

[1 mark]

(b) write down the inverse of the element *a*, where

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

[1 mark]

(c) demonstrate that the operation of composition is associative using the following elements

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

[2 marks]

(d) Explain why it is possible for the group S_4 to have a subgroup of order 4. You do not need to find such a subgroup.

[2 marks]

Question 5

Further A-Level, Pure Mathematics 2, June 2024, Paper 4A, Q7 (Edexcel)

The set of matrices $G = \{\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ where

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

with the operation $\otimes_2$ of matrix multiplication with entries evaluated modulo 2, forms a group.

(a) Show that $\mathbf{B}$ is an element of order 3 in G

[2 marks]

(b) Determine the orders of the other elements of G

[3 marks]

(c) Give a reason why *G* is **not** isomorphic to

(i) a cyclic group of order 6

(ii) the group of symmetries of a regular hexagon

[2 marks]

The group *H* of permutations of the numbers 1, 2 and 3 contains the following elements, denoted in two-line notation,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(d) Determine an isomorphism between the groups *G* and *H*

[3 marks]

Question 6

Given $\mathbb{Z}_2 = \{0,1\}$, the Cartesian product

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$$

has *four* elements.

This set inherits a binary structure via addition of co-ordinates

$$(x, y) + (v, w) := (x + v, y + w)$$

where $x + v$ and $y + w$ are both computed in $(\mathbb{Z}_2, +_2)$.

This binary operation has an addition table that should look very familiar: it has exactly the same structure as the Klein four-group!

+	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

$\cong$

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We conclude that $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong K_4$ is indeed a group

(a) List the elements of the Cartesian product of $(\mathbb{Z}_3, +_3)$ and $(\mathbb{Z}_2, +_2)$

[3 marks]

(b) Determine $\langle(1,1)\rangle$

[2 marks]

(c) Use your answers to parts (a) and (b) to deduce which group $\mathbb{Z}_3 \times \mathbb{Z}_2$ is isomorphic to.

[2 marks]