3.1 Coupled Systems

By definition, a coupled system comprises of two first order differential equations with two dependent variables and one independent variable. Typically, t is the independent variable and represents time. The dependent variables, x and y, each depend of t. In biology x and y may represent the populations of two species of animal that have a symbiotic relationship. For example, one may be a predator and the other, prey. The population of each depends both on how large their own population is, and the size of the other's.

Coupled System Solution Technique

Coupled first-order linear differential equations may be solved by eliminating one of the dependent variables to form a second-order differential equation. This is then solved using methods previously studied.

3.2 Example

(i) Find the general solutions to the differential equations,

$$\frac{dx}{dt} = x + 5y \qquad \frac{dy}{dt} = -x - 3y$$

Work on first equation: $\frac{dx}{dt} = x + 5y$

$$5y = \frac{dx}{dt} - x \qquad \text{Making } y \text{ the subject}$$

$$y = 0.2\frac{dx}{dt} - 0.2x$$

$$\frac{dy}{dt} = 0.2\frac{d^2x}{dt^2} - 0.2\frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\frac{dy}{dt} = -x - 3y$

$$0.2\frac{d^2x}{dt^2} - 0.2\frac{dx}{dt} = -x - 3\left(0.2\frac{dx}{dt} - 0.2x\right)$$

$$= -x - 0.6\frac{dx}{dt} + 0.6x$$

$$0.2\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + 0.4 = 0$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2 = 0$$

Next, write down and solve the auxiliary equation...

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(4)}}{2}$$

$$= -1 \pm \mathrm{i}$$

$$x = e^{-t}(A\cos t + B\sin t)$$

Differentiate this as we need a "*y* =" companion to this "*x* ="

$$\frac{dx}{dt} = e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t)$$

$$= e^{-t}((B - A)\cos t + (-A - B)\sin t)$$

Go back to the first equation: $\frac{dx}{dt} = x + 5y$

$$e^{-t}((B - A)\cos t + (-A - B)\sin t) = e^{-t}(A\cos t + B\sin t) + 5y$$

$$5y = e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

$$y = \frac{1}{5}e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

The general solutions are,

$$x = e^{-t}(A\cos t + B\sin t)$$

$$y = \frac{1}{5}e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

[8 marks]

(ii) Given that at time $t = 0$, $x = 1$ and $y = 2$, find the particular solutions

$$x = e^{-t}(A\cos t + B\sin t)$$

$$1 = A$$

$$y = \frac{1}{5}e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

$$2 = \frac{1}{5}(B - 2A)$$

$$10 = B - 2$$

$$B = 12$$

The particular solutions are,

$$x = e^{-t}(\cos t + 12\sin t)$$

$$y = e^{-t}(2\cos t - 5\sin t)$$

[4 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

(i) Find the general solutions to the differential equations,

$$\frac{dx}{dt} = -x + 6y \qquad\qquad \frac{dy}{dt} = x - 2y$$

[8 marks]

(ii) Given that at time $t = 0$, $x = 2$ and $y = 0$, find the particular solutions

[4 marks]

Question 2

A tank of water on O'Kevin's farm contains two different types of chemical that react with each other. The rates of change of each chemical can be modelled using the following differential equations;

Photograph by Tracy's Photography

$$\frac{dx}{dt} = -3x + 2y$$

$$\frac{dy}{dt} = -2x + y$$

In the equations, x is the number of litres of chemical X and y is the number of litres of chemical Y at time t hours.

Initially there is 100 litres of chemical X and 200 litres of chemical Y in the tank.

(i) Show that the solutions to the differential equations can be written as, $x = P e^{-t}$ and $y = Q e^{-t}$ where P and Q are functions of t to be found.

[8 marks]

(ii) Find, correct to three significant figures, the amount of each chemical in O'Kevin's tank at t = 2 hours.

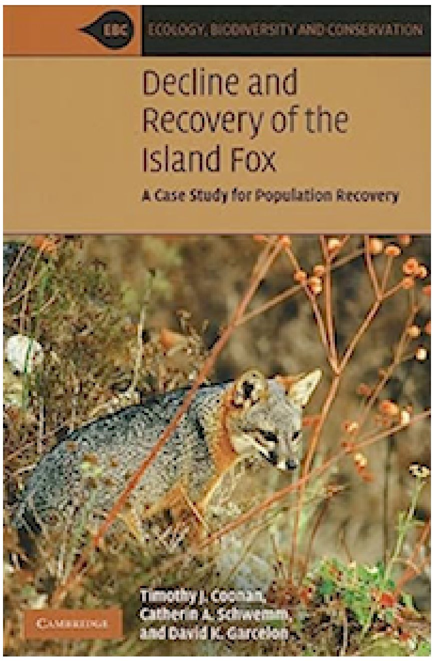
[2 marks]

(iii) Use the model to describe what happens to the amount of each chemical in O'Kevin's tank as t gets large.

[2 marks]

Question 3

At the start of 2023, Professor Oscar Workfast, and his rival Professor Boris Beach initiated an annual survey on the number of Weasels, x, and Island Foxes, y, on one of the Channel Islands of California. The Weasels preys on the Island Fox. The mathematical model they are developing involves the following two differential equations;



$$\frac{dx}{dt} = 0.2x + 0.2y$$

$$\frac{dy}{dt} = -0.5x + 0.4y$$

(i) Show that $\dfrac{d^2x}{dt^2} - 0.6\dfrac{dx}{dt} + 0.18x = 0$

[3 marks]

(ii) Prof Workfast claims that the general solution for the number of weasels at time t is $x = e^{\alpha t}\,(A \cos \beta t + B \sin \beta t)$.
Show that this is correct and find the values of the constants α and β.

[4 marks]

(iii) Not to be outdone, Prof BB claims that the number of Island Foxes will be an equation of the form $y = P\,e^{\alpha t}(Q \cos \beta t + R \sin \beta t)$.
Show that this is also correct and find the value of the constant P, and determine the functions Q and R in terms of A and B.

[3 marks]

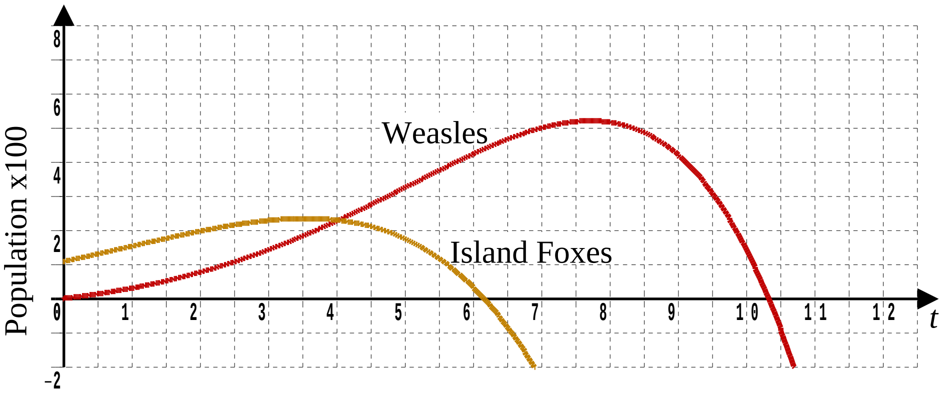
(iv) Prof Beach loves Island Foxes and despises the Weasels that prey upon them. Prof Workfast worships the Weasels and views the Island Foxes as nothing more than food for his Weasels. In the survey, the professors count 3 Weasels and 111 Island Foxes.
During which year does the model predict the Island Foxes die out ?

[5 marks]

(v) How many Weasels will there be when the Island Foxes die out ?

[1 mark]

(vi) The particular solutions to the model are graphed below.



With reference to the graph, comment on the model.

[2 marks]