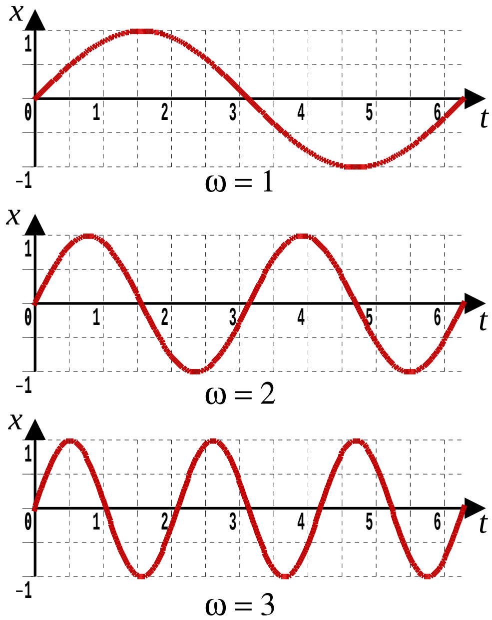
2.1 Period and Frequency



The graphs above are of the oscillations $y = \sin(\omega t)$ for $\omega = 1$, $\omega = 2$ and $\omega = 3$.

The value of ω gives the number of oscillations within 2π radians of time.

Definition : Period of a Function

The period, P, of an oscillatory function is the amount of time for the graph to complete a single oscillation.

$$P = \frac{2\pi}{\omega}$$

Example

So for the above three graphs the periods are,

$$\text{for } y = \sin t \text{ the period is } \frac{2\pi}{1} = 2\pi$$

$$\text{for } y = \sin 2t \text{ the period is } \frac{2\pi}{2} = \pi$$

$$\text{for } y = \sin 3t \text{ the period is } \frac{2\pi}{3}$$

Definition : Frequency of a Function

The frequency, f, of an oscillation is the number of vibrations per second.

$$f = \frac{1}{P} \quad \text{where } P \text{ is the period}$$

2.2 Forced Harmonic Motion

Simple harmonic motion is characterised by the differential equation,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

and the frequency associated with this equation is that with which the system it describes would naturally vibrate. This is termed the system's resonant frequency. A system can be made to vibrate at a rate that is different to its natural rate in which case the above equation may take the form,

$$\frac{d^2x}{dt^2} + \omega^2 x = f(t)$$

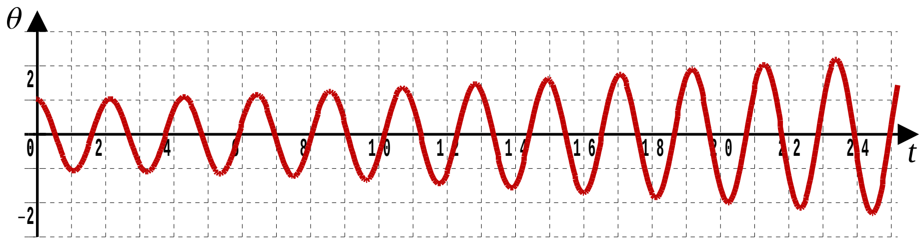
This is termed forced harmonic motion and the last question of the previous lesson (Exercise 1.2, Question 5) featured a system undergoing forced harmonic motion. The system being studied was that of a simple pendulum swinging from side to side where θ was the amount of angle, in radians, away from the vertical downward as time varied. The equation looked at was,

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2} \cos 3t$$

This would naturally vibrate at a rate of one third of an oscillation per second and the waveform would be a (phase shifted) sine wave. In this case the forcing term was in tune with the natural rate and so the waveform remained essentially a (phase shifted) sine wave but the forcing term is feeding in extra energy into the system and so the oscillations are increasing in amplitude. In fact, in answering the question it was worked out that,

$$\theta = \frac{\pi}{3} \cos 3t + \frac{1}{12} t \sin 3t$$

This is graphed below and the main feature to note is that the swing of the pendulum is steadily increasing. This is a system that may eventually break !



2.3 Damped Harmonic Motion

The foregoing illustrated a system that would eventually shake itself apart; an undesirable property ! The model for simple harmonic motion can be refined by adding an additional force which is proportional to the velocity of the particle. When this force acts such that it slows the particle it is known as a damping force and the motion of the particle is termed damped harmonic motion. This will have a second order differential equation of the form,

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0 \quad \text{for positive constants } k \text{ and } \omega^2$$

where x is the displacement from a fixed point at time, t.

The nature of the damping is classified as being one of three types, depending on the discriminant, D, of the auxiliary equation.

$\boldsymbol{D > 0}$: two distinct real roots; the system experiences **heavy damping**
The system does not oscillate. Motion is curtailed as if moving through treacle.

$\boldsymbol{D = 0}$: one repeated root; the system experiences **critical damping**
The system does not oscillate and motion is briskly curtailed.

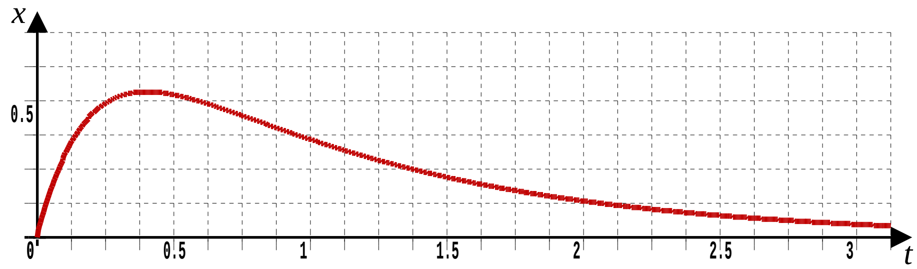
$\boldsymbol{D < 0}$: two complex roots; the system experiences **light damping**
The system oscillates with an amplitude that decreases exponentially over time.

2.4 Example of Heavy Damping

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 5x = 0 \quad \text{and when } t = 0,\ x = 0 \text{ and } \frac{dx}{dt} = 4$$

This has two distinct roots and so is an example of heavy damping.

This has general solution, $x = e^{-t} - e^{-5t}$ graphed below.



The push-tap in many motorway service stations, restaurants and schools is an example of a heavily damped mechanism. The tap is pressed and water flows for around half a minute whilst the tap slowly returns by itself to the off position.

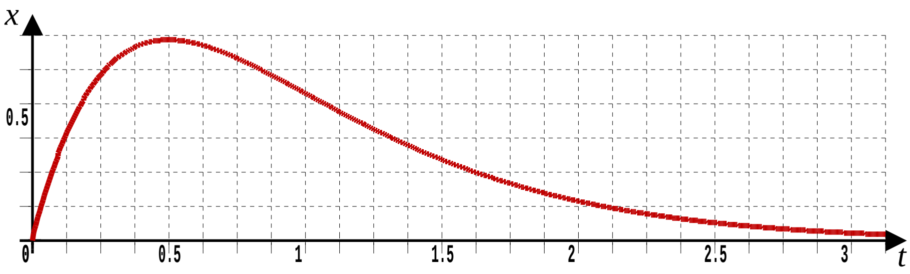
Their main advantage is that they can not be left running to overflow the basin either accidentally or by anti-social behaviourists.

2.5 Example of Critical Damping

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0 \quad \text{and when } t = 0,\ x = 0 \text{ and } \frac{dx}{dt} = 4$$

This has a repeated root and so is an example of critical damping.

This has general solution $x = 4t\,e^{-2t}$ graphed below.



A door closer is an example of a critically damped system. For fire safety and energy conservation reasons, the door should automatically close in as short a time as possible without slamming.

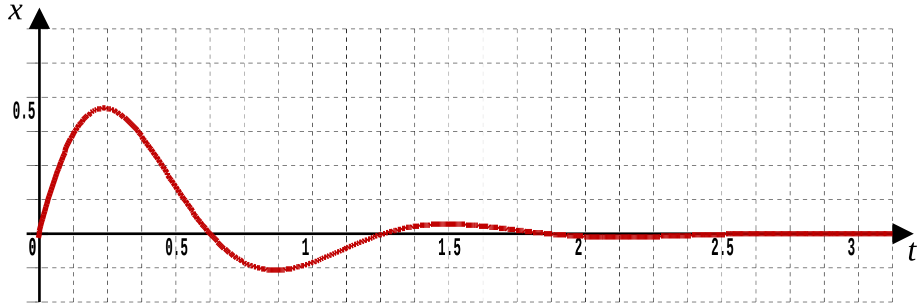
A ship's compass is another example of a critically damped device. When the ship changes direction the compass has to point in the new direction as quickly as possible without oscillating about the new heading. The damping is provided by filling the plastic sphere housing the device with clear oil.

2.6 Example of Light Damping

$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 29x = 0$ and when $t = 0,\ x = 0$ and $\frac{dx}{dt} = 4$

This has complex roots and so is an example of light damping.

This has general solution $x = \frac{4}{5}e^{-2t}\sin 5t$



If an electric light, handing from a ceiling by a wire, is accidentally knocked it will swing to and fro for a while. Slowly, over time, it will swing with less and less amplitude until is is again stationary. This is indeed “light” damping !

2.7 Undamped Simple Harmonic Motion

What is often referred to as simply “simple harmonic motion”, characterised by,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is **undamped** simple harmonic motion.

The amplitude of its oscillations remain constant over time.

2.8 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 84

Question 1

Caspian × Herring Gull photograph by Matt Livesey, December 2019

A seagull is resting in calm weather conditions on a buoy on a lake.
When disturbed, it flies off.
The subsequent motion of the buoy is modelled by the differential equation,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

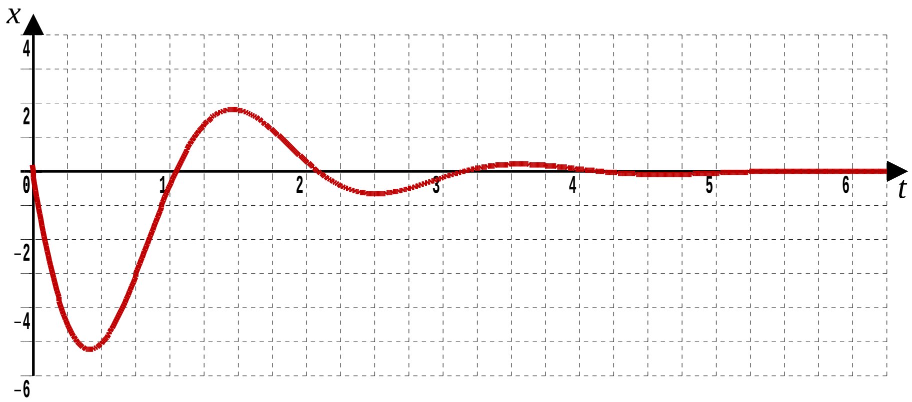
where t is time measured in seconds, and x is the vertical displacement, in metres, of the waterline of the buoy above or below the surface of the lake (up is positive).

When $t = 0$, $x = 0$ and, from the force of the seagull taking off, $\frac{dx}{dt} = -\frac{1}{4}$ m.s^{-1}

(i) Find x as a function of t

[6 marks]

A graph of the buoy's motion is given below, with displacement, x, given in centimetres (rather than metres) and time, t, in seconds.



(ii) Is the motion heavily, critically or lightly damped ?
Give a reason for your answer.

[1 mark]

(iii) Find the magnitude of the maximum displacement under the lake's surface of the buoy's waterline during the motion.
Give your answer in centimetres to one decimal place.

[5 marks]

Question 2

Further A-Level Examination Question from June 2019, Paper 2, Q6 (Edexcel)

An engineer is investigating the motion of a sprung diving board at a swimming pool. Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.

A diver jumps from the diving board.

The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation,

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

[2 marks]

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 $cm.s^{-1}$

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.

[8 marks]

(c) Comment on the suitability of the model for large values of t

[2 marks]

Question 3

Further A-Level Examination Question from June 2021, Paper 1, Q6 (Edexcel)

A tourist decides to do a bungee jump from a bridge over a river.

One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.

The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is 0 m.s^{-1}
- the acceleration of the tourist is 13.6 m.s^{-2}

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$5k\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + 17x = 0$ for $t \geqslant 0$ and where k is a positive constant.

(a) Determine the value of k

[2 marks]

(b) Determine the particular solution to the differential equation.

[7 marks]

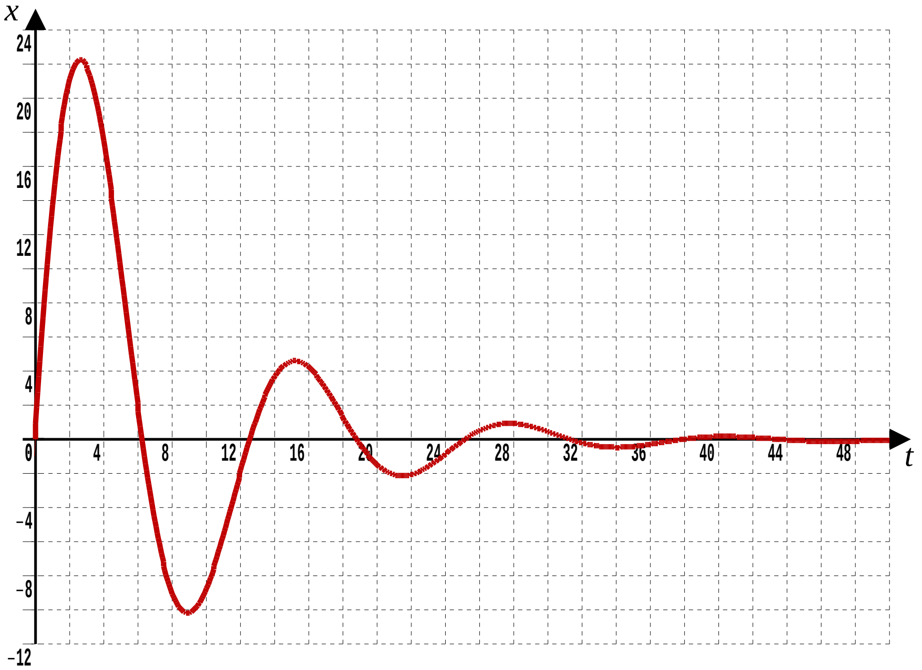
(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point.

[2 marks]

(d) Give a limitation of the model.

[1 mark]

Question 4



Graphed is lightly damped harmonic motion of the form $x = Ae^{-kt} \sin(\omega t + \alpha)$ with displacement, x, in metres and time, t, in seconds.

(i) Explain why $\alpha = 0$ radians.

[1 mark]

(ii) What is the period, P, the frequency, f, and the angular velocity, ω, of the waveform ?

[3 marks]

Question 5

Further A-Level Examination Question from May 2020, Paper 1, Q3 (Edexcel)

A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination. The concentration of antibodies, x, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation $100\dfrac{d^2x}{dt^2} + 60\dfrac{dx}{dt} + 13x = 26$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

[4 marks]

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

[8 marks]

A second dose of the vaccine has to be given to try to ensure that it is effective. It is only safe to give the second dose if the concentration of the antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

[2 marks]

Question 6

A particle P is attached to one end of a light elastic spring. The other end of the spring is fixed to a point A on the smooth horizontal surface on which P rests. The particle is held at rest with AP = 0.9 metres and then released. At time t seconds the displacement of the particle from A is x metres. The motion of the particle can be modelled using the equation,

$$\ddot{x} = -200x$$

(i) State the type of motion exhibited by the particle P.

[1 mark]

Given that $x = 0.3$ and the particle is at rest when $t = 0$

(ii) solve the differential equation to find x as a function of t

[7 marks]

(iii) find the period and amplitude of the motion

[3 marks]

(iv) calculate the maximum speed of *P*

[2 marks]

Question 7

Further A-Level Examination Question from June 2009, FP2, Q8 (Edexcel)

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$$

Given that $x = 0$ and $\dfrac{dx}{dt} = 2$ at $t = 0$,

(a) find x in terms of t

[8 marks]

The solution to part (a) is used to represent the motion of a particle P on the x-axis. At time t seconds, where $t > 0$, P is x metres from the origin O.

(b) Show that the maximum distance between O and P is $\dfrac{2\sqrt{3}}{9}$ metres and justify that this distance is a maximum.

[7 marks]