Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

APPLICATIONS OF DIFFERENTIAL EQUATIONS

A pendulum can be a lot of fun !

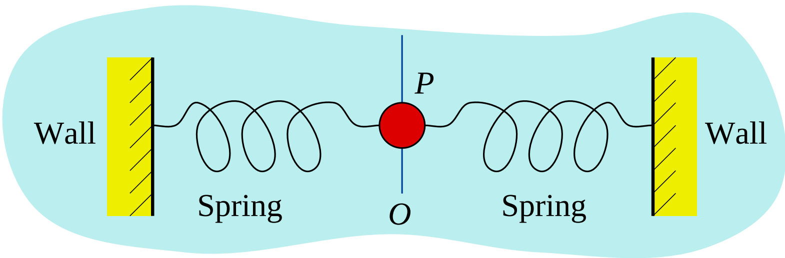
The Sea Dragon at Fantasy Island Amusement Park, New Jersey, USA

A P P L I C A T I O N S
O F
D I F F E R E N T I A L
E Q U A T I O N S

Lesson 1

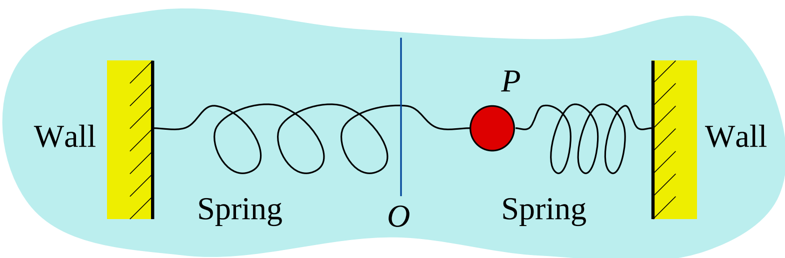
1.1 Simple Harmonic Motion

The classic mechanical example of a system that exhibits simple harmonic motion (SHM) is illustrated below.



Shown is a bird's eye view of a particle, *P*, resting on ice and sitting on the midway line, *O*, between a pair of parallel walls with two identical elastic springs attached.

The next illustration shows the particle after it has been moved and held to one side of *O*. It is now released and oscillates about the central position, *O*. This idealised oscillatory motion (that ignores any friction effects) is described as simple harmonic.



Simple harmonic motion is characterised by having an acceleration (towards *O*) that is proportional to the displacement of *P* from *O*. Written in mathematics this becomes $\frac{d^2x}{dt^2} = -\omega^2 x$ where ω^2 is the constant of the proportionality.

Physicists call ω the angular velocity of the particle. Note that the minus sign is there because the acceleration is always directed towards *O*, the centre of the oscillation. The above equation can be written as $\frac{d^2x}{dt^2} + \omega^2 x = 0$ which is a second order homogeneous differential equation. Number Wonder's Differential Equations II covers how to solve such equations.

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

A particle is moving to and fro along a straight line.
At time t seconds its displacement in metres, x, from a fixed point O is such that,

$$\frac{d^2x}{dt^2} = -9x$$

At $t = 0$, $x = 1$ and the particle is moving with a velocity of 4 $m.s^{-1}$

(i) Is the motion of the particle simple harmonic ?

[1 mark]

(ii) By the use of standard techniques to obtaining the particular solution of a second order differential equation, find an expression for the displacement of the particle after t seconds.

[5 marks]

(iii) By writing your answer in the form $x = R\sin(\omega t + \alpha)$ determine the maximum displacement of the particle from O and the first time, correct to 3 significant figures, that the particle attains this maximum.

[4 marks]

Question 2

A particle P is moving along a straight line.
At time t seconds, the acceleration of the particle is given by

$$a = t + 5v, \ t \geqslant 0$$

(i) Show that the situation described can be modelled by the first order differential equation $\frac{dv}{dt} - 5v = t$

[1 mark]

(ii) Is the motion of the particle simple harmonic ?
Give a reason for your answer.

[2 marks]

(iii) State the integrating factor associated with the differential equation.

[1 mark]

(iv) Given that $v = 0$ when $t = 0$, show that the velocity of the particle at time t is given by an equation of the form $v = A(Be^{kt} - Ct - 1)$ where A, B, C and k are positive constants to be found.

[5 marks]

Question 3

A particle moves along a straight line. The particle moves such that its acceleration, in m.s^{-2}, acts towards a fixed point *O* and is proportional to its distance, *x* m, from *O*.

(i) Describe the motion of the particle.

[1 mark]

Given that the acceleration of the particle is -5 m.s^{-2} when $x = 1$,

(ii) write down a differential equation to describe the motion of the particle.

[2 marks]

If the velocity and displacement of the particle at time $t = 0$ are 6 m.s^{-1} and 5 m,

(iii) find an expression for the displacement of the particle after *t* seconds.

[7 marks]

(iv) Hence find the maximum distance of the particle from *O*.

[2 marks]

Question 4

Further A-Level Examination Question from June 2011, FP2, Q8 (Edexcel)

The differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \cos 3t, \quad y \geqslant 0$

describes the motion of a particle along the x-axis.

(a) Find the general solution of this differential equation.

[8 marks]

(b) Find the particular solution for which, at $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$

[5 marks]

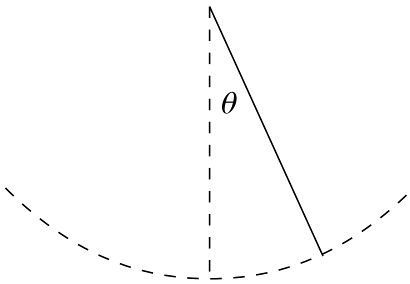
On the graph of the particular solution defined in part (b), the first turning point for $t > 30$ is the point A.

(c) Find approximate values for the coordinates of A.

[2 marks]

Question 5

Further A-Level Examination Question from June 2022, Paper 1, Q10 (Edexcel)



The motion of a pendulum is modelled by the differential equation,

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

where θ is the angle, in radians, that the pendulum makes with the downward vertical t seconds after it begins to move.

(a) **(i)** Show that a particular solution of the differential equation is,

$$\theta = \frac{1}{12} t \sin 3t$$

[4 marks]

(ii) Hence, find the general solution of the differential equation.

[4 marks]

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of α radians with the downward vertical,

(b) determine, according to the model, the value of α to 3 significant figures.

[4 marks]

Given that the true value of α is 0.62

(c) evaluate the model.

[1 mark]

The differential equation $\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2} cos\, 3t$ models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

[1 mark]