2.1 A Mathematical Description

Drawing a flowchart to describe *The Collatz Rule* is time consuming and takes up an excessive amount of space on the page. A more succinct description of the iteration at the heart of the Collatz conjecture is this;

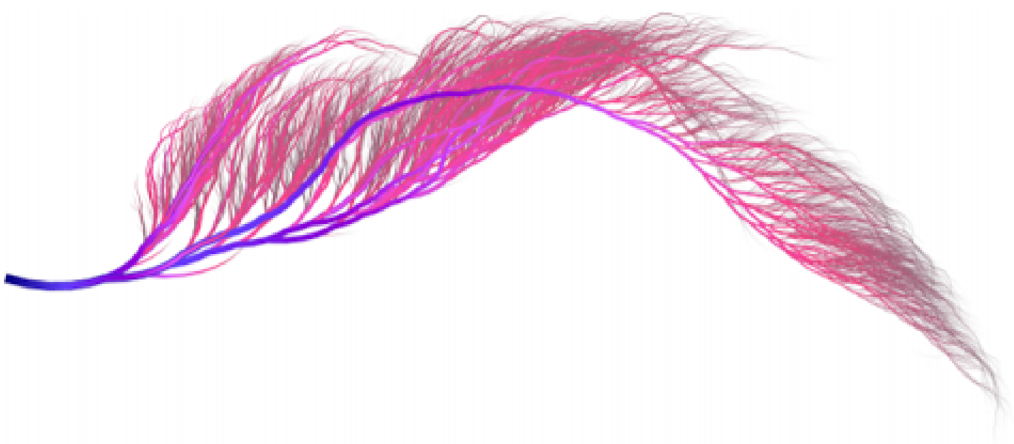
$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

Our strategy to begin constructing the Collatz tree has been to pick a positive integer that is not in our existing tree and work out its "fall to earth". Here is the *Collatz 100 Table* that we had worked out by the end of the previous lesson,

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The shaded numbers are those for which we know the Collatz conjecture is true.

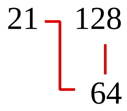
We can be more certain of getting the whole tree by working our way up the tree, starting from the 1 at the base. Computer visualisations of the tree, such as the one below, known as the feather, use this *grow from the base* approach.



An algorithm that will let us move up instead of down is needed. Developing this, which is attended to next, is a little tricky although the end result is pleasingly simple to apply.

2.2 Grow from the Base

The Collatz tree often has a junction where two branches meet. For example, as we have just seen, at the number 64 there is the following;



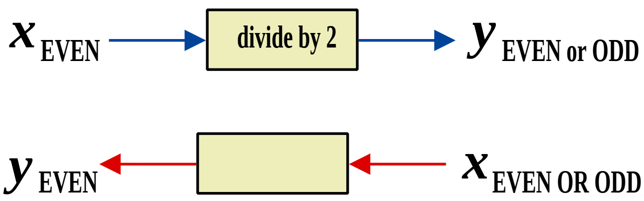
This raises questions such as,

- Do all numbers end up being where two branches meet ?
- Can we, instead of flowing down the tree, flow up ?

Having the mathematical description of the tree is the key to answering these questions. *The Collatz Rule* is,

$$y = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

To find a reversing rule for the evens, consider going through the following diagram “the other way”;

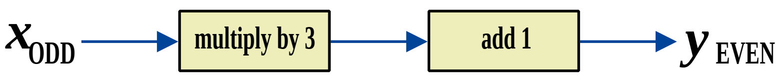


$y_{REVERSE} =$

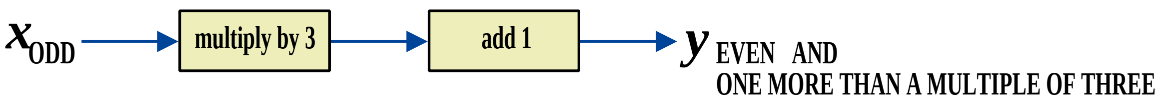
[2 marks]

Finding a reversing rule for the odds is more tricky.

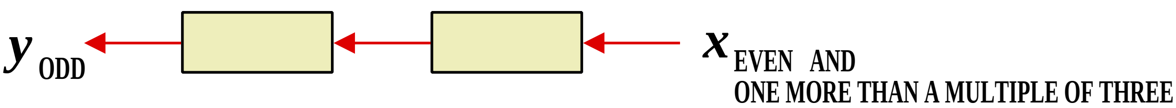
To make progress, consider going through the odd rule as before where we multiply by 3 and add 1. What will come out will always be even.



Furthermore, what comes out is one more than a multiple of three.



So, for example, 10 is even and also one more than a multiple of 3 but 8, although even, is not one more than a multiple of three. So when going in reverse it's fine to put 10 in but not 8.



$y_{REVERSE} =$

[3 marks]

2.3 The Reversed Collatz Rule With Examples

Many animations of the Collatz conjecture start from the number one at the base of the tree and draw the tree going upwards, rather than starting at a randomly chosen starting number the drawing the part of the tree below. *The Reversed Collatz Rule* is the algorithm of how to do this.

The Reversed Collatz Rule

One reverse always occurs and is given by,

$$y_{REVERSE} = 2x$$

Additionally, if x is even and one more than a multiple of three, then there is a branch in the tree and a second reverse which is given by,

$$y_{REVERSE} = \frac{x - 1}{3}$$

Example #1 : Reversing 7

14
|
7

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 7 = 14$

As 7 is odd there is no second reverse.

Example #2 : Reversing 64

21 ┐ 128
|
64

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 64 = 128$

As 64 is even so ask yourself, is it also one more than a multiple of three ?

It is because 64 divided by 3 has remainder one third (remainder 0.33333...)

There is a second reversal $y_{REVERSE} = \frac{x - 1}{3} = \frac{64 - 1}{3} = \frac{63}{3} = 21$

Example #3 : Reversing 44

88
|
44

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 44 = 88$

As 44 is even so ask yourself, is it also one more than a multiple of three ?

It is **NOT**: 44 divided by 3 has remainder two thirds (remainder 0.66666...)

There is no second reverse.

2.4 Exercise

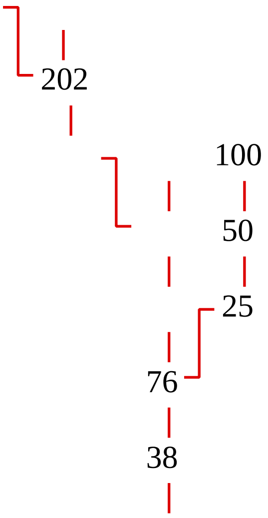
Marks Available : 22 plus 8 bonus

Question 1

(i) Find the two reversals of 202.

[2 marks]

(ii) Use your part (i) answers to help complete the following piece of tree.



[3 marks]

Question 2

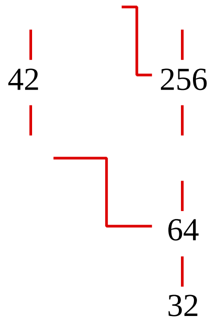
(i) Find all reversals of 42.

[2 marks]

(ii) Find all reversals of 256.

[2 marks]

(iii) Use your part (i) and (ii) answers to help complete this piece of tree;



[3 marks]

Question 3

Mini-Project (15 minutes)

This question is about reversing up the tree beyond the high point reached in this exercise's first question, where 67 was reached.

Starting from 67 begin reversing up the tree.

Explore all branches but stop reversing on a particular branch if any number greater than or equal to 400 occurs.

Draw the tree to illustrate your discoveries.

Update the *Collatz 100 Table* as you progress.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[10 marks]

Question 4 (Bonus)

Write a computer program that takes a positive integer as input and writes out the path that number takes as it "falls to earth".

[8 bonus marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2024 Number Wonder
Detailed worked solutions to the exercises are available from m.hamsen2@unimail.derby.ac.uk