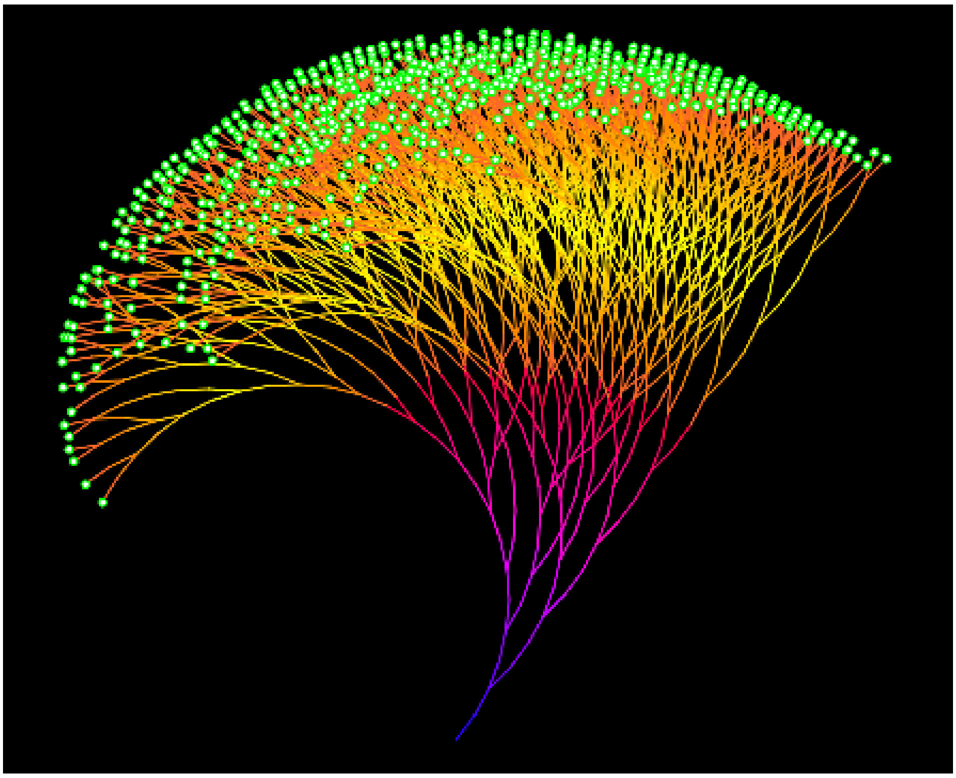
A-Level Mathematics Project
The Collatz Conjecture

Academic Perspectives

THE COLLATZ CONJECTURE

The easiest to understand unsolved problem in mathematics

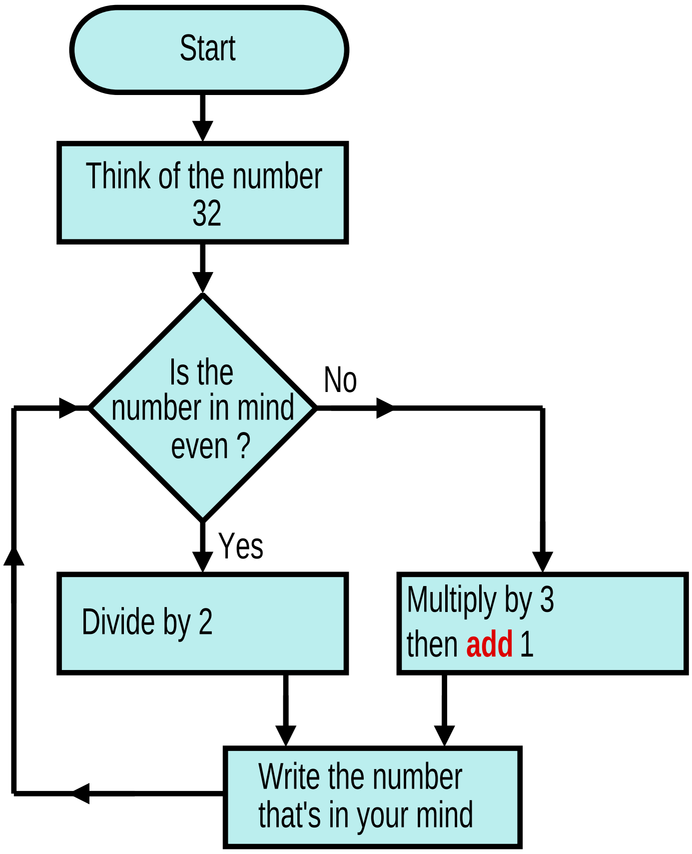


1.1 An Introduction To The Collatz Conjecture

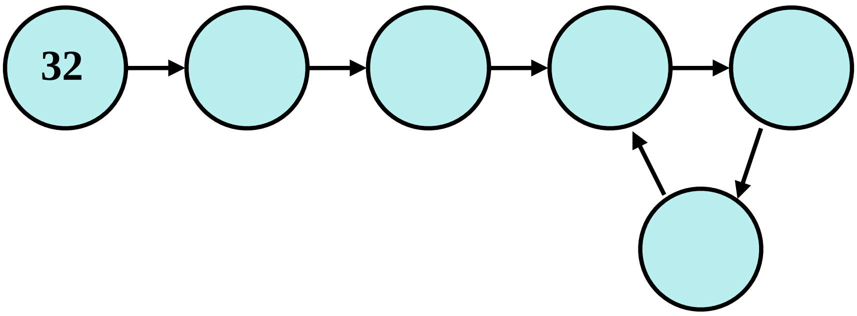
Is the image is of coral seaweed ? Or blood vessels ? Or the neural network of some small creature ? In fact it's a beautiful representation of a sequence of numbers generated from the Collatz conjecture. A conjecture is a mathematical statement that is believed by the person making it to be true, but it may not be. It's an invitation to look into something that another mathematician considers interesting. The conjecture is named after the German mathematician Lothar Collatz (1910-1990) who formulated it in 1937. Simple to state and understand, to this day not known if the Collatz conjecture is true or false. The reputation of the conjecture is such that if you were to settle the matter, one way or the other, you would immediately become famous, not just in the world of mathematics, but across front pages of newspapers around the globe. We are going to take a look at the conjecture, and others like it, and perhaps get to the point where we can produce interesting and colourful representations of the number sequences involved, like the one above.

1.2 The Iteration At The Heart of the Collatz Conjecture

An iterative process is one in which you repeatedly carry out the same set of instructions. The idea of iteration has been around for over two hundred years but only since the invention of the desktop computer in the late 1970s has it become mainstream mathematics. A flowchart can be used to describe an iteration.



On the following diagram write out the numbers generated by the flowchart.

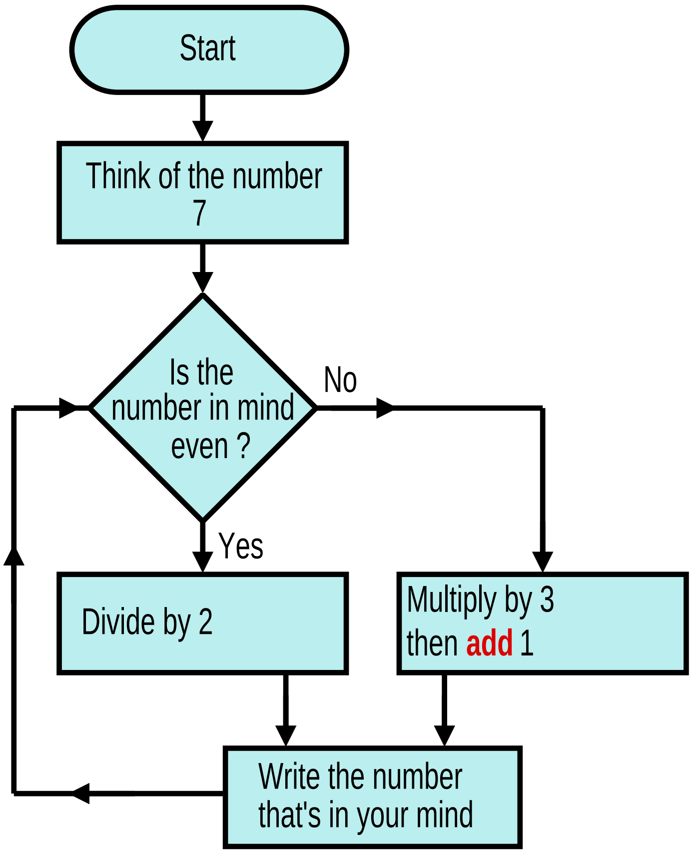


[2 marks]

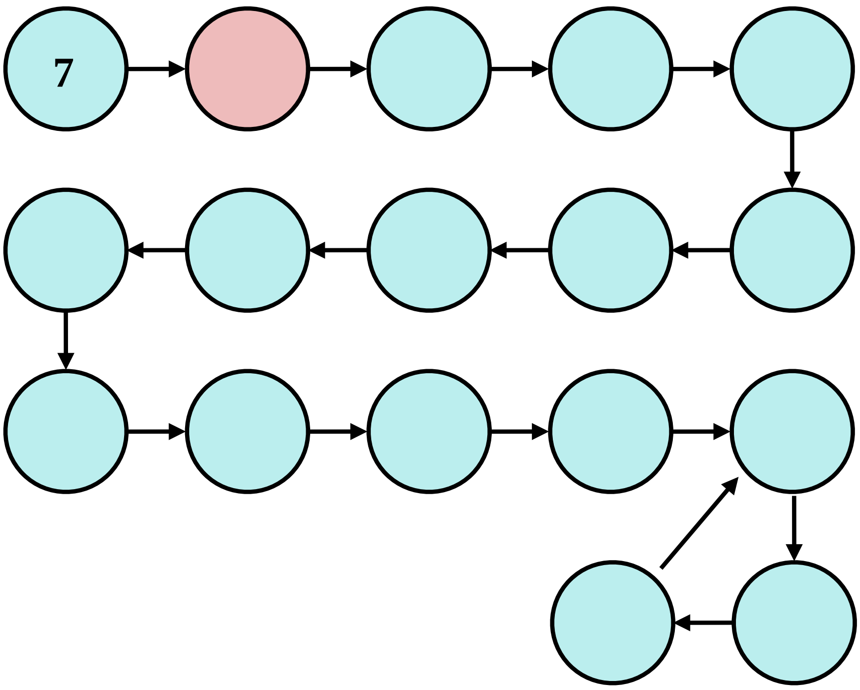
The iteration described in the flowchart shows that, if you start with the number 32, the sequence eventually reaches the number 1. You could, of course, start with a different positive integer. Would the same iteration rule, starting from this different number, also pass through the number 1 ?

Let's begin to investigate by starting with the number 7.
(The 7 is my choice; you'll get to chose your own starting number soon enough!)

1.3 Starting From Seven



On the following diagram write out the numbers generated starting from 7.



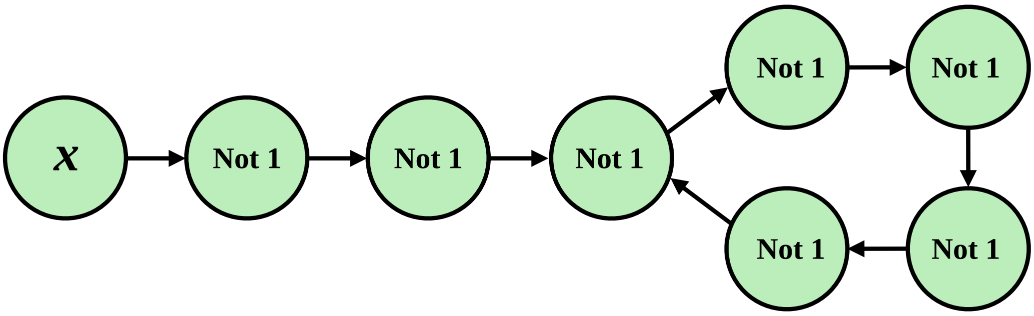
[3 marks]

1.4 The Collatz Conjecture

The Collatz Conjecture claims that, ***no matter what positive integer you start with***, the resulting number sequence will always eventually pass through the number one. Some sequences, like that starting from seven, increase and decrease many times but the claim is that, all the sequences will “fall to earth”, meaning that eventually the number one will occur. To this day no one has found a counterexample of a starting number for which this does not happen, but nor can anyone find a logical explanation for why it will always happen.

1.5 The Search For A Counterexample

The most straight forward way to prove that the Collatz conjecture is false is to find a counterexample. The counterexample would be a starting number that perhaps causes the terms in the resulting sequence to increase and decrease several times before going into a loop, all without involving the number one. So, for example, starting with *x*, the number path could look something like this;



Generally speaking, when faced with a conjecture, it's always worth initially trying to find a counterexample. Trying to prove that something is true, when in fact it is false, is a good way to waste a lot of time. This is because mathematicians are primarily interested in showing that conjectures are true. The following joke illustrate this;

My Mathematical Week

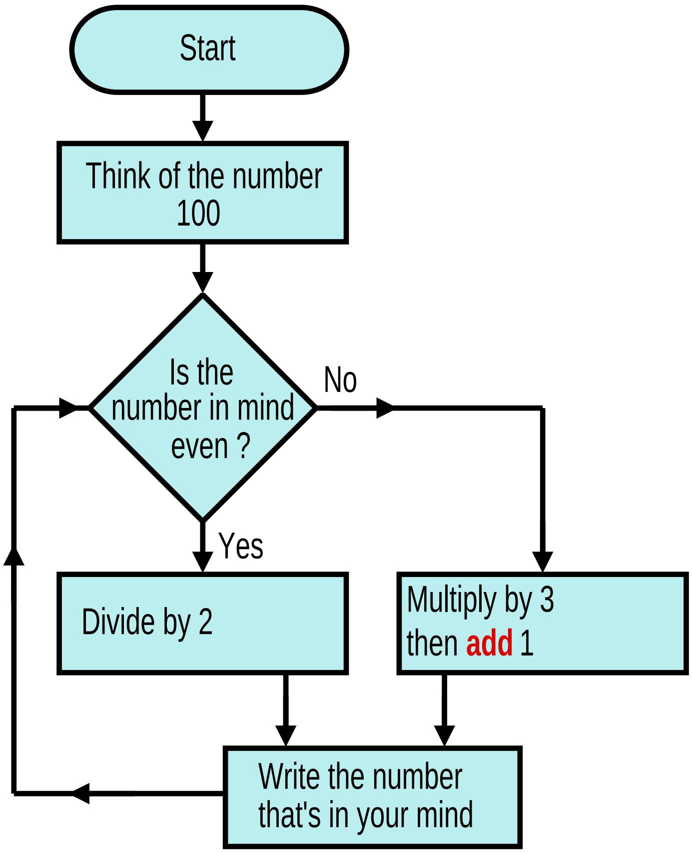
Monday: tried to prove my conjecture is true
Tuesday: tried to prove my conjecture is true
Wednesday: tried to prove my conjecture is true
Thursday: tried to prove my conjecture is true
Friday: realised that my conjecture is false

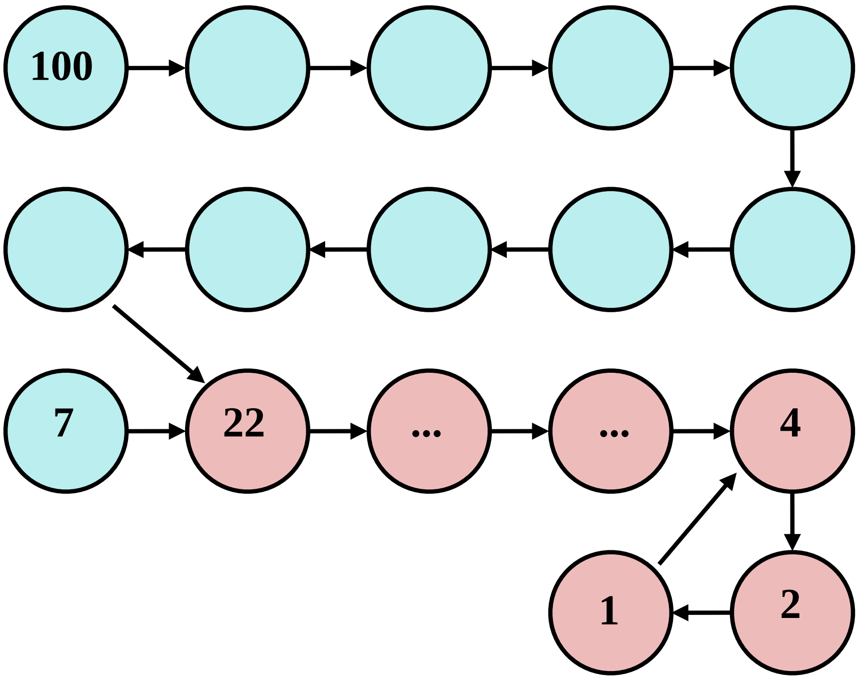
1.6 An Efficient Counterexample Search

Although you are, no doubt, keen to get started on trying out your own starting number, let's look at a situation that will illustrate how we can massively increase the efficiency of our search for a counterexample.

The basic idea behind an efficient search is this; From the work just done, you already know what happens to many other starting numbers. For example, in the red circle of the search that started with 7, you have the number 22. This means that you already know what happens if 22 is used as the start number. So, the only numbers worth trying now are numbers that did not occur in any circles of a previous answer.

There is one other useful observation, best illustrated by an example;





[2 marks]

Notice that as soon as any number occurred that could be found in a previous diagram you knew how the story would end. So when the number 22 occurred, there was no need to do any more work; you knew how the path then made its way to the terminating loop of [4, 2, 1].

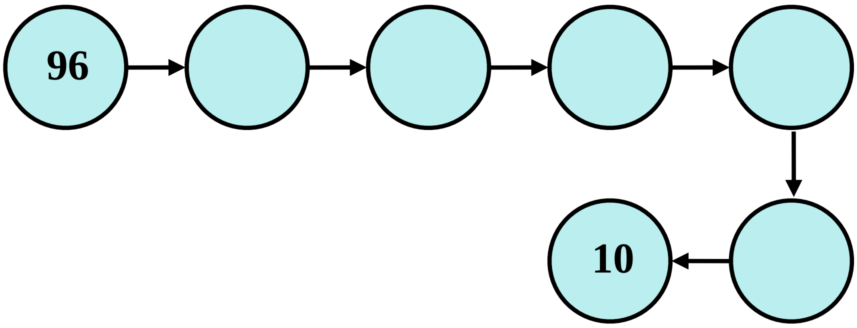
1.7 Exercise

Marks Available : 30

This exercise is about adding a numbers in the correct places in the Collatz tree on the adjacent page and updating the *Collatz 100 Table* at every opportunity.

Question 1

(i) Apply the Collatz iterative rule to the number 96, stopping at 10.



[1 mark]

(ii) Why was there no need to proceed once the number 10 was reached ?

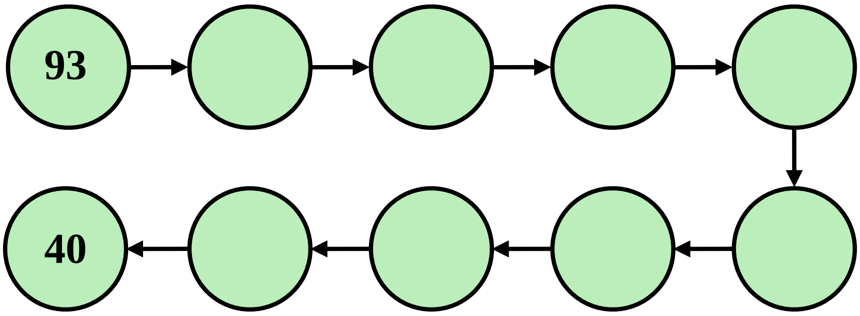
[1 mark]

(iii) Write these on the tree on the adjacent page in the appropriate place.
Update the *Collatz 100 Table* (also on the adjacent page).

[2 marks]

Question 2

(i) Apply the Collatz iterative rule to the number 93, stopping at 40.



[2 marks]

(ii) Write these on the tree on the adjacent page in the appropriate place.
Update the *Collatz 100 Table* (also on the adjacent page).

[2 marks]

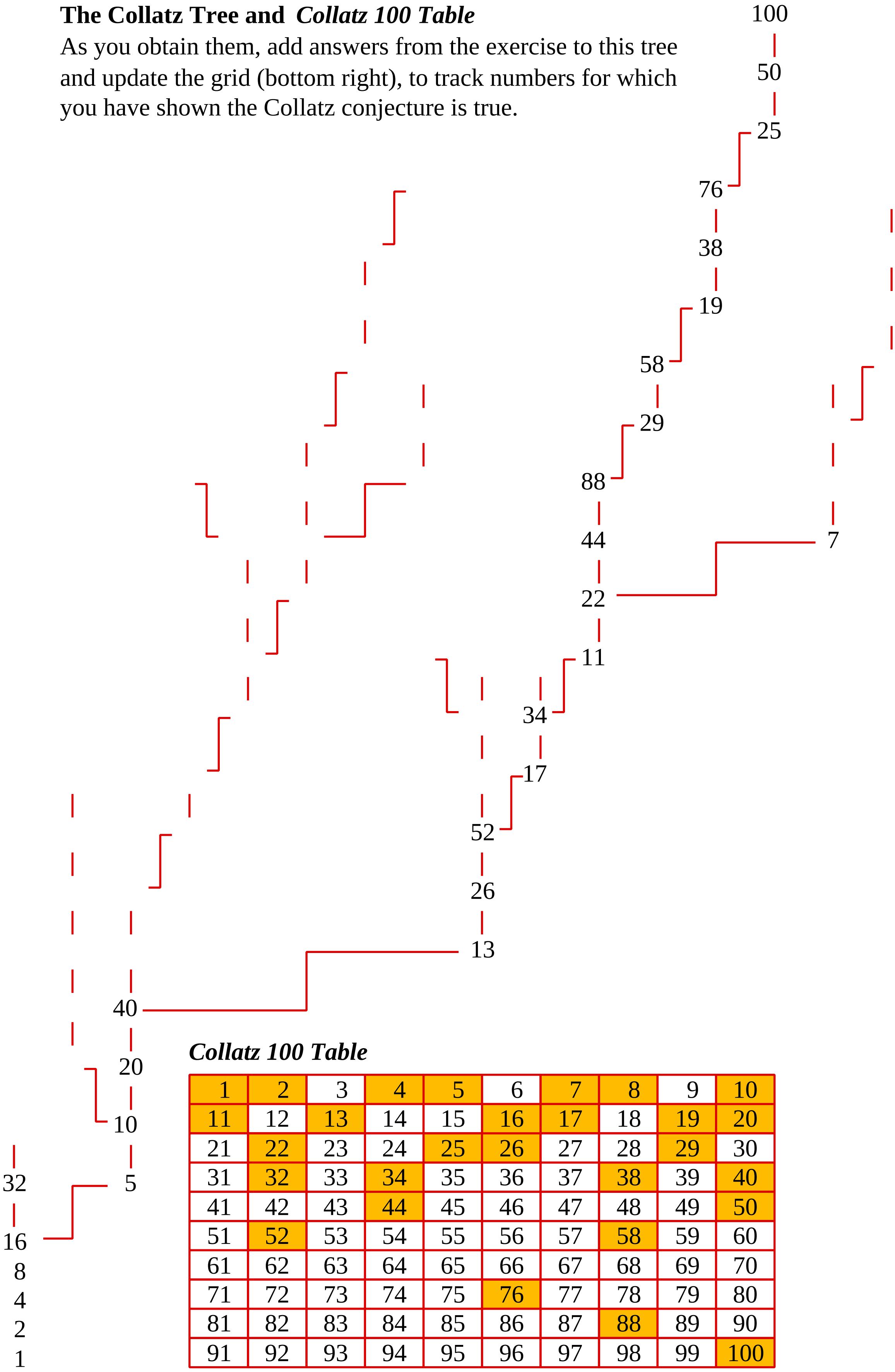
Question 3

One part of working backwards to go up the tree instead of down is easy.
The number directly above any number is double that number.
Work backwards to add a number above 32 on the tree.
Update the *Collatz 100 Table*.

[1 mark]

The Collatz Tree and *Collatz 100 Table*

As you obtain them, add answers from the exercise to this tree and update the grid (bottom right), to track numbers for which you have shown the Collatz conjecture is true.



Collatz 100 Table

1	**2**	3	**4**	**5**	6	**7**	**8**	9	**10**
11	12	**13**	14	15	**16**	**17**	18	**19**	**20**
21	**22**	23	24	**25**	**26**	27	28	**29**	30
31	**32**	33	**34**	35	36	37	**38**	39	**40**
41	42	43	**44**	45	46	47	48	49	**50**
51	**52**	53	54	55	56	57	**58**	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	**76**	77	78	79	80
81	82	83	84	85	86	87	**88**	89	90
91	92	93	94	95	96	97	98	99	**100**

Question 4

(i) Work backwards to add above the 52 on the previous page, the number 104, then, above that, 208.

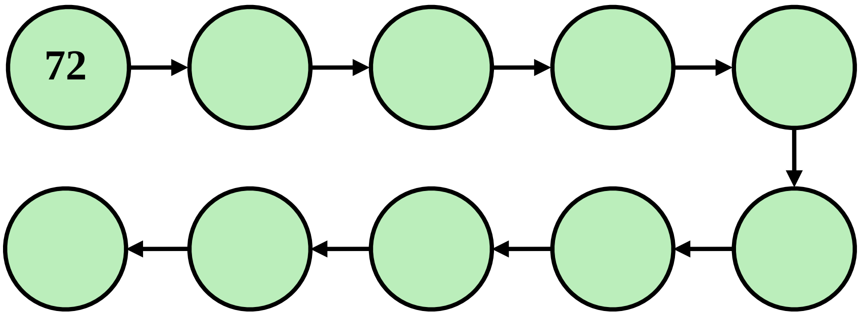
[1 mark]

(ii) To continue going backwards (up) you need to deal with a branch.
We'll look at how to tell if there is a branch or not in the next lesson.
There are two numbers above the 208.
What are these two numbers ?
Write them on the tree on the previous page.
Update the *Collatz 100 Table*.

[2 marks]

Question 5

(i) Come down the tree, starting with 72.
After a short while a number already in the tree is encountered.
There are more circles than you need on the diagram !
(To see if you spot when to stop)



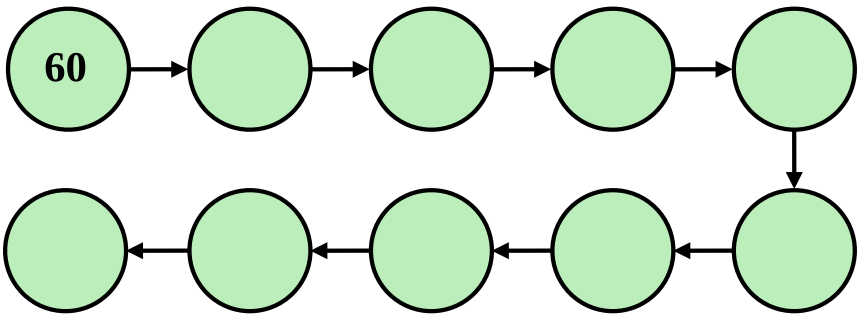
[2 marks]

(ii) Update the tree and the *Collatz 100 table* on the previous page.

[2 marks]

Question 6

(i) Come down the tree, starting with 60.
After a short while a number already in the tree is encountered.
There are more circles than you need on the diagram !
(To see if you spot when to stop)



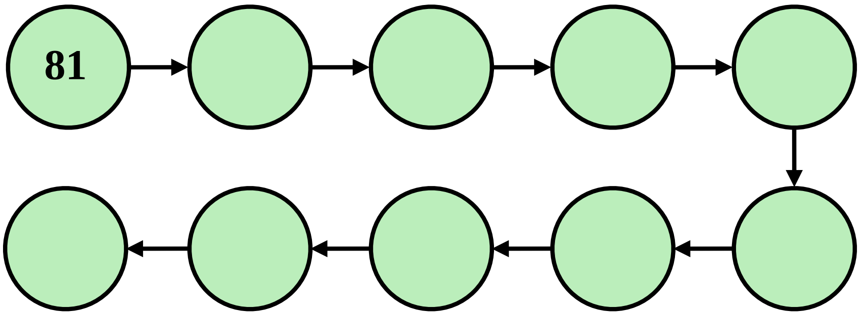
[2 marks]

(ii) Update the tree and the *Collatz 100 table* on the previous page.

[2 marks]

Question 7

(i) Come down the tree, starting with 81.
After a short while a number already in the tree is encountered.
There are more circles than you need on the diagram !
(To see if you spot when to stop)



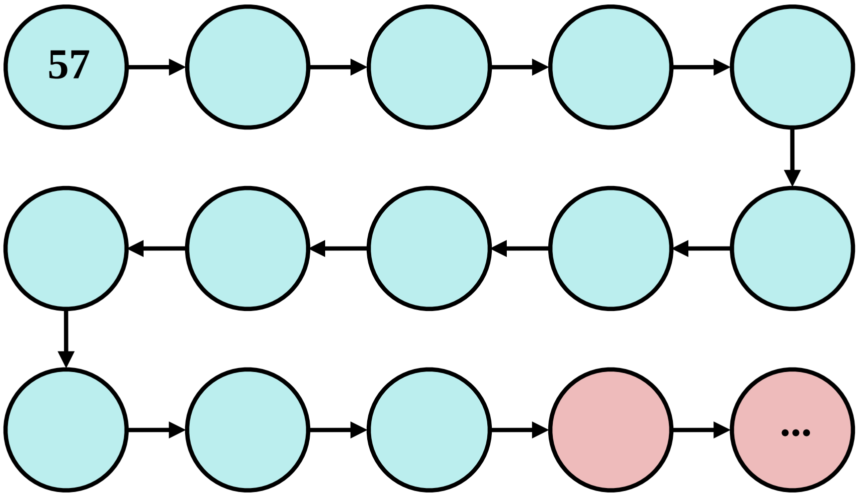
[2 marks]

(ii) Update the tree and the *Collatz 100 table* two pages back.

[2 marks]

Question 8

(i) Come down the tree, starting with 57.
After a while a number already in the tree is encountered.



[3 marks]

(ii) What problem would be encountered if you tried to add this sequence onto the tree we have been building over the last seven questions ?

[1 mark]

(iii) This sequence gives you eight more numbers to add to the *Collatz 100 table* of those numbers for which you know the conjecture is true.
Update the *Collatz 100 table* two pages back.

[2 marks]