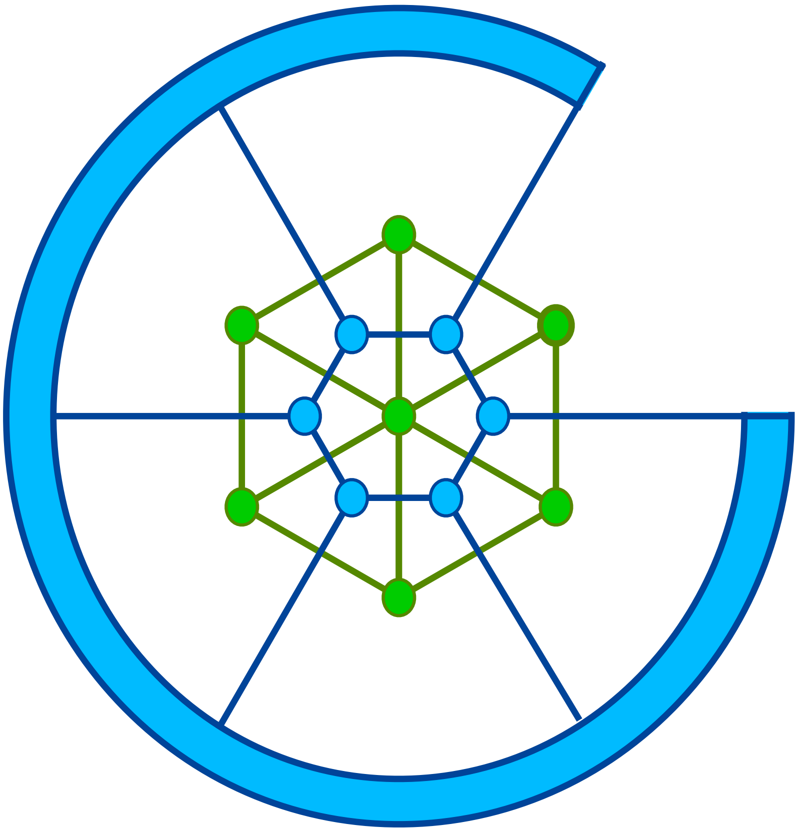
Undergraduate Lectures in Mathematics

Graph Theory I

A Guide for Beginners

Martin Hansen



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Graph Theory I : A Guide for Beginners
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Cover Image

The cover is unconventional in that the graph depicted has a vertex that's been smeared into an arc in order to show in a more symmetrical way a well known graph and its dual. In green is the graph of the wheel, ω_6, and, in blue, its dual.

About this work

These lecture notes are for an third year undergraduate course on Graph Theory. They have a focus on the aspects of the topic that interest me and which, I hope, makes for a lively, colourful and somewhat “different” presentation. Assumed are the basics of Matrix algebra, although key ideas are briefly revisited. I have included comprehensive solutions to the exercises as, to me, a part of what makes mathematics interesting is the fine detail. I enjoy developing and perfecting robust solutions to sets of problems.

In writing these lectures I consulted a wide variety of sources. In particular I have consulted extensively the powerpoint presentations given to undergraduates at the University of Derby by Dr Nicholas Korpelainen, Professor Robin Wilson's classic book, “Introduction to Graph Theory”, now in its fifth edition, alongside the related Open University's MT365 course on the subject. Many of the questions in the exercises are my own, often variations on established themes; several are what might be termed “the classics” of the subject.

Martin Hansen
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January 2023

G R A P H   T H E O R Y   I

Lecture 1

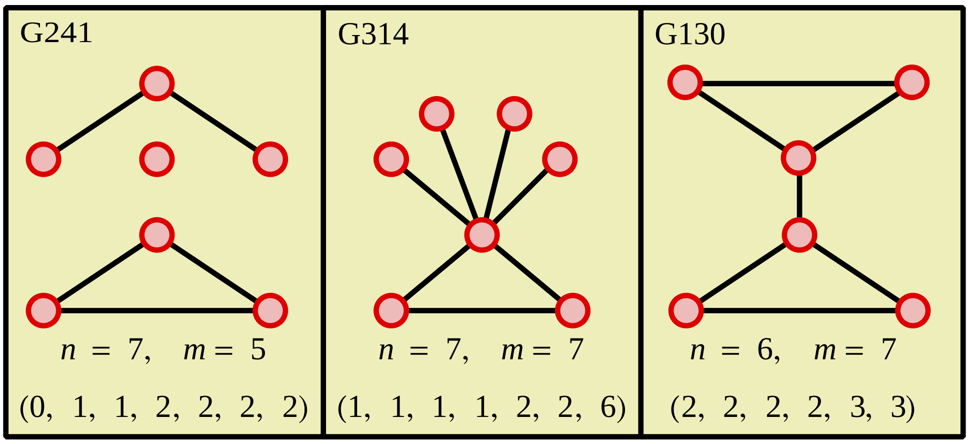
Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

1.1 What is a Graph ?

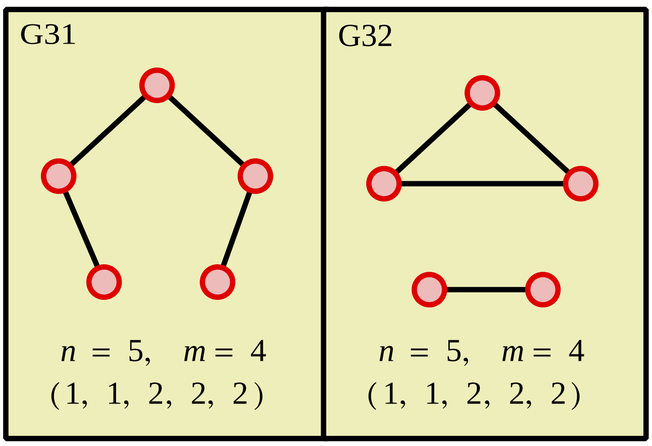
A graph is an ordered pair of sets, $G = (V(G), E(G))$, where the elements of V and E are called *vertices* and *edges* respectively. Each edge is associated with two distinct vertices and is said to join them. As beginners, our interest is in *simple* graphs where, at most, there can be one edge between any vertex pair.

1.2 Visualisation of Graphs

Three examples of graphs are given below. Various catalogues of small simple graphs have been made over the years and the numbers G241, G314 and G130 refer to a catalogue from The Open University. Modern catalogues can be found online in the form of searchable databases. G241 is an example of a graph that is *disconnected*, whilst G314 and G130 are *connected*.

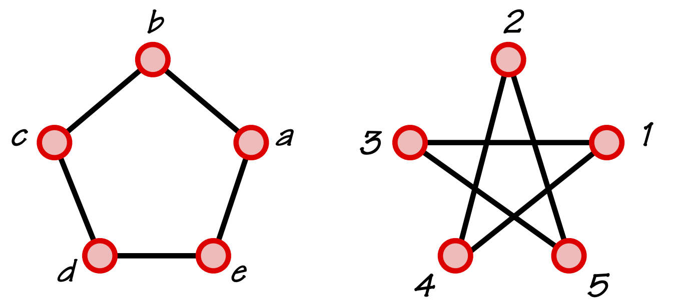


The number of vertices of a graph G is usually denoted by n and is termed the *order* of the graph. The number of edges, m, is the *size* of G. The number of edges that meet at a vertex gives the *degree* of that vertex. The degrees of all the vertices of a graph may be listed as a monotonic increasing sequence, one for which $a_k \leqslant a_{k+1}$ for all positive integer k less than n. Two topologically distinct graphs may have the same *degree sequence* as G31 and G32 demonstrate.



1.3 Some Graph Descriptors

If it is helpful to do so each vertex of a graph may be labelled distinctly, typically using either letters or numbers.

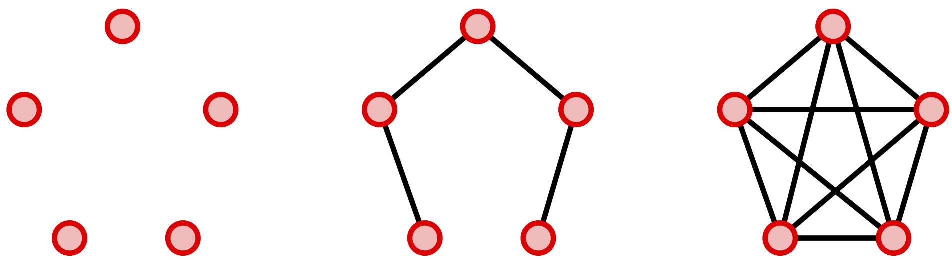


Two labelled graphs G and H are *isomorphic* if there is a bijection f (a one-to-one and onto function) between $V(G)$ and $V(H)$ such that $\{v, w\} \in E(G)$ if and only if $\{f(v), f(w)\} \in E(H)$. Intuitively, isomorphism identifies when two seemingly different graphs, such as the two above, have the same underlying structure. To see that the two graphs above are isomorphic notice that each connects the five vertices as if they are the beads on a loop of string. If required, an isomorphism can be explicitly stated as being, for example, in this case,

$$f(a) = 1,\ f(b) = 3,\ f(c) = 5,\ f(d) = 2 \text{ and } f(e) = 4$$

Note that "isomorphic to" defines an equivalence relation on any set of labelled graphs. It partitions such a set into equivalence classes, called isomorphism classes. Any catalogue of graphs will only give one example from an isomorphism class.

The above two graphs are both a cycle on 5 vertices, C_5. This is an example of a *cycle graph*, a graph that consists of a single cycle of vertices and edges and denoted C_n, $n \geqslant 3$. The graph C_5 is also an example of a *regular graph*, one where all the vertices have the same degree, in this case 2. It could be described as being 2-regular. A regular graph is *complete* if each vertex is joined to each of the others by exactly one edge, and denoted K_n. A regular graph with no edges is termed a null graph, N_n. A path on $n > 2$ distinct vertices is written P_n, and will have two end vertices of degree at least one and thread its way through each of the remaining $(n - 2)$ vertices which will each be of degree at least 2.



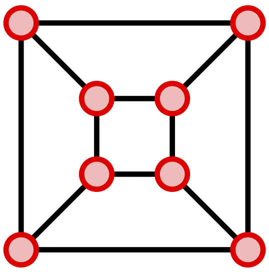
N_5 P_5 K_5

1.4 Polyhedra

Many everyday objects, polyhedral in shape, can be modelled straight forwardly as graphs. The photographs below are of a waste paper basket. On the left the basket is upside down, showing that it is, topologically speaking, equivalent to a cube. On the the right it is photographed directly from above.

Photographs by Martin Hansen

Of the two views, the photograph to the right translates most easily into a graph, and the result is given below.



What makes this graph the best representation of a cube is the fact that it is uncluttered to look at and this is due in part to the fact that it is a *planar* graph; it can be presented without edges being drawn over one another. The edges of the graph divide the plane into regions called the *faces* of the plane graph. These regions are all bounded but for one, that one being termed the *infinite* or *external* face. The *length* of a face is the number of edges bounding it.

Theorem 1.1 : Euler's Polyhedral Formula

If a connected plane graph has n vertices, m edges and f faces then,

$$n - m + f = 2$$

1.5 Reasoning Why Euler's Polyhedral Formula is True

For the cube, the associated planar graph has 8 vertices, 12 edges and 6 faces, and Euler's polyhedral formula holds in this case. More than simply observing that it holds for a cube, we can look into why it holds. We'll do this in a way that will readily generalise to show that the same formula it holds for all polyhedra that do not have any holes in them.

The key idea is to work with the cube's planar graph and identify some operations that reduce the complexity of that graph whilst leaving invariant the value of $n - m + f$.

1.5.1 The Triangularisation Move

Pick a face, if any, with more than three sides. Add an edge across that face. Notice that this adds one edge and one face to the graph but leaves the value of $n - m + f$ unchanged because,

$$n - (m + 1) + (f + 1) = n - m + f$$

1.5.2 Face Removal Move

Look for a face, if any, which shares precisely one edge with the exterior face. Remove this face by removing the one shared edge. Notice that this subtracts one edge and one face from the graph but does not alter the value of $n - m + f$ because,

$$n - (m - 1) + (f - 1) = n - m + f$$

1.5.3 Vertex Removal Move

Look for a face, if any, which shares precisely two edges with the exterior face. Remove this face by removing both these shared edges and their shared vertex. Notice that this subtracts two edges, one vertex and one face from the graph. For the new graph we have,

$$(n - 1) - (m - 2) + (f - 1) = n - m + f$$

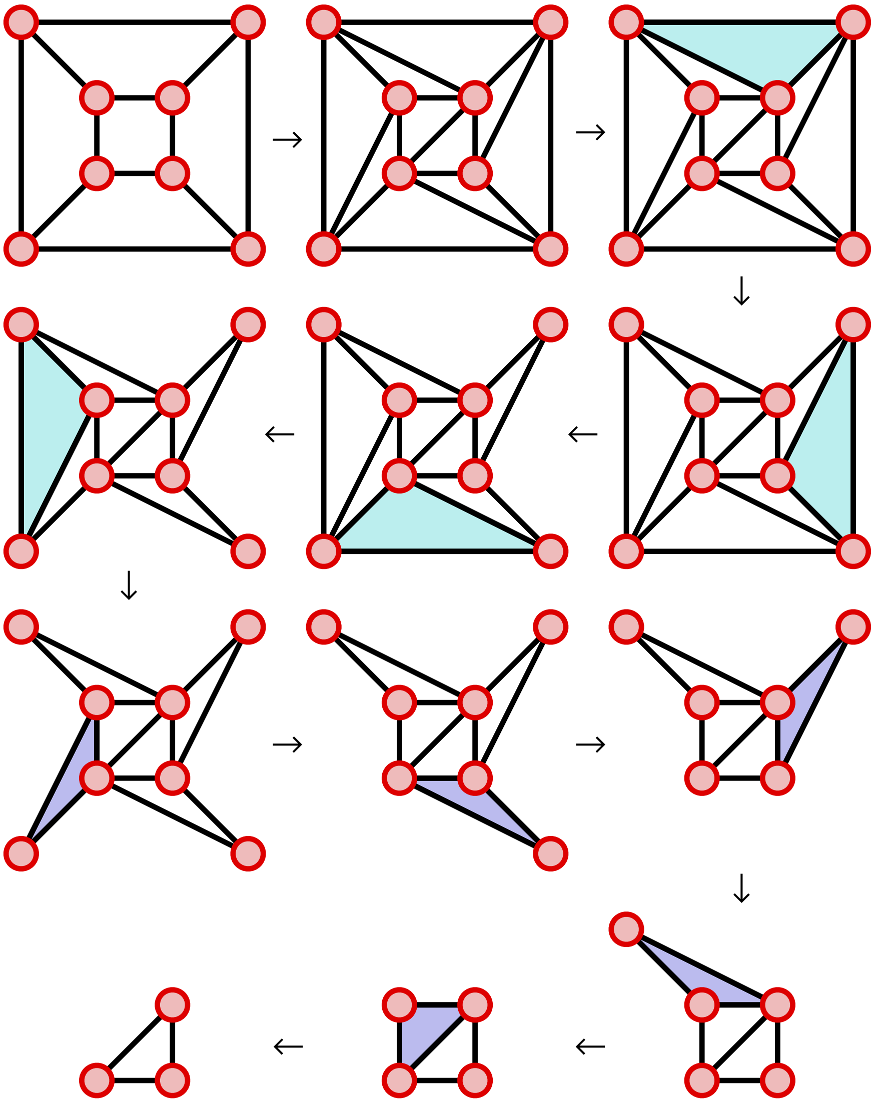
showing that, once again, the value of $n - m + f$ is unaltered by the move.

1.5.4 Applying the Moves

Apply the triangularisation move repeatedly until the entire graph has been triangularised. Always apply the vertex removal rule if it is possible to do so (To prevent the graph fragmenting and becoming disconnected). When the vertex removal rule cannot be performed apply the face removal rule once before switching back to always applying the vertex removal rule if it is possible to do so. Continue in this manner until it becomes impossible to do so.

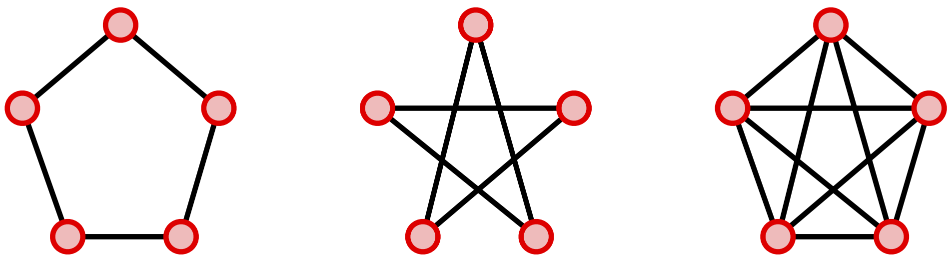
The algorithm will eventually stop with the terminating graph being that of a single triangle for which $n = 3$, $m = 3$ and $f = 2$ (don't forget the external face). For this triangle, $n - m + f = 2$, and so this must also have been true for the original planar graph and associated polyhedron.

The next diagram shows the process being applied to the planar graph associated with a cube.



1.6 The Complement of a Graph

The complement of a graph G is a graph $\overline{G}$ with the same vertex set V but whose edge set E consists of the edges not present in G. The graph sum $G + \overline{G}$ on graph of degree n is therefore the complete graph K_n.
A graph G is *self-complementary* if $G = \overline{G}$



C_5 + $\overline{C_5}$ = K_5

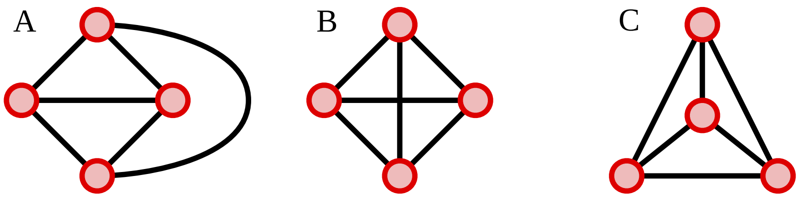
As an example, C_5 and its complement $\overline{C_5}$ are shown. Their graph sum is K_5. Previously it was noted that the two graphs being summed were isomorphic. Thus C_5 is an example of a self-complementary graph.

1.7 Exercise

Marks Available : 80

Question 1

The following three graphs are three representations of the complete graph K_4



(i) Write down the degree sequence for K_4

[1 mark]

(ii) Which one of the three graphs is not planar ?

[1 mark]

(iii) K_4 is an example of a regular graph.
Is it 3-regular, 4-regular or 6-regular ?

[1 mark]

(iv) How many faces has graph C ?

[1 mark]

(v) Which of the five platonic solids could be represented by K_4 ?

[1 mark]

Question 2

Draw all eleven possible unlabelled simple graphs with four vertices.
Under each write its degree sequence.

[6 marks]

Question 3

(i) Provide a proof of the following theorem,

Theorem 1.2 : The Handshaking Lemma
Suppose that a graph G has n vertices and m edges, with degree sequence given by $(d_1, d_2, ..., d_n)$. Then $\sum_{1}^{n} d_i = 2m$

[4 marks]

(ii) Explain why The Handshaking Lemma is so called.

[2 marks]

(iii) Using The Handshaking Lemma, prove by contradiction that;

Lemma 1.1 : The Even Number of Odds
Every graph has an even number of odd-degree vertices.

[5 marks]

Question 4

A connected graph is one in which there is a path from any point to any other point on the graph. A graph that is not connected is said to be disconnected. Connected graphs arise naturally when derived from a polyhedron.

(i) Let G be a 3-regular connected planar graph with 20 vertices.
Determine the number of regions (faces) in the graph.

[4 marks]

(ii) Let G be a p-regular connected planar graph with n vertices.
Prove that either the number of vertices or the regularity must be even.

[4 marks]

Question 5

Prove by induction;

Lemma 1.2 : Edge Count of a Complete Graph (A Clique)

The complete graph K_n has exactly $\dfrac{n(n-1)}{2}$ edges

[6 marks]

Question 6

(i) Prove that there are no 3-regular graphs with five vertices.

[3 marks]

(ii) Prove that, if n and r are both odd, then there are no r-regular graphs with n vertices.

[3 marks]

Question 7

Prove the following lemma;

Lemma 1.3 : Vertex Pair Of Equal Degree

In any finite simple graph with at least two vertices, there must be at least two vertices which have the same degree.

[4 marks]

Question 8

Let G be a graph with nine vertices such that each vertex is of either degree 5 or of degree 6. Show that G has at least six vertices of degree 5, or at least five vertices of degree 6.

[4 marks]

Question 9

Prove that, to be disconnected, a graph on n vertices can have, at most, a number of edges, m, that is given by,

$$m = \frac{(n-1)(n-2)}{2}$$

[6 marks]

Question 10

(i) Prove that a set of x elements has 2^x subsets.

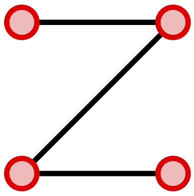
[3 marks]

(ii) Show that there are exactly $2^{\frac{1}{2}n(n-1)}$ labelled simple graphs on n vertices.

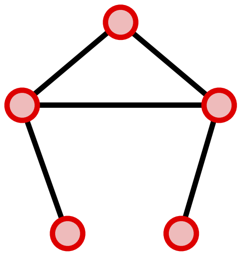
[3 marks]

Question 11

(i) Draw the complements to each of the following graphs.
Under each write its degree sequence.



(1, 1, 2, 2)



(1, 1, 2, 3, 3)

[2 marks]

(ii) A graph G on n vertices is self-complementary with degree sequence,

$$(d_1, d_2, ..., d_n)$$

Determine the relationship between d_i and d_{n+1-i}

[2 marks]

(iii) Use your part (ii) relationship to generate a list of the degree sequences, one of which any self-complementary graphs on five vertices must satisfy.

[4 marks]

(iv) Show that, if a graph G is isomorphic to its complement, then the number of vertices of G has the form $4k$ or $4k + 1$ for $k \in \mathbb{Z}^+$

[5 marks]

(v) Furthermore, if G is regular, show that $n \equiv 1 \pmod 4$

[5 marks]

1.8 Answers to Exercise 1.7

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Answer 1

(i) The degree sequence for K_4 is (3, 3, 3, 3)

(ii) No edges cross on a planar graph.
Graph B is not planar.

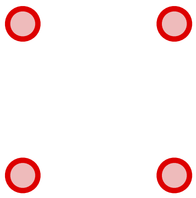
(iii) At degree of each vertex is three and so K_4 is 3-regular.

(iv) Graph C has four faces (don't forget the external face).

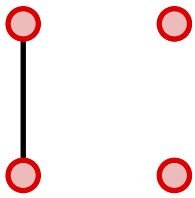
(v) K_4 could represent a tetrahedron.

[5 marks]

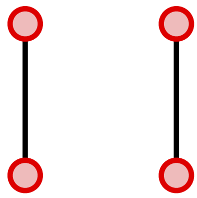
Answer 2



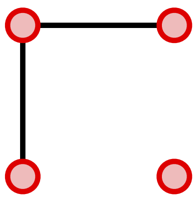
(0, 0, 0, 0)



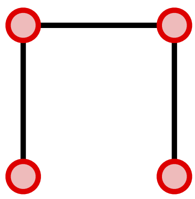
(0, 0, 1, 1)



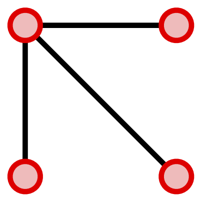
(1, 1, 1, 1)



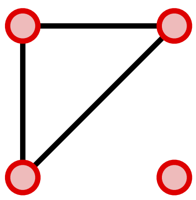
(0, 1, 1, 2)



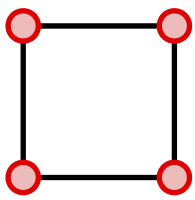
(1, 1, 2, 2)



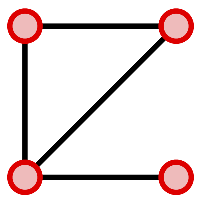
(1, 1, 1, 3)



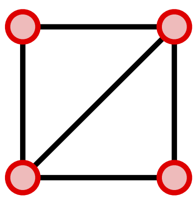
(0, 2, 2, 3)



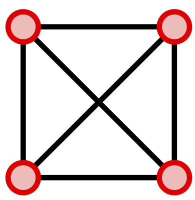
(2, 2, 2, 2)



(1, 2, 2, 3)



(2, 2, 3, 3)



(3, 3, 3, 3)

[6 marks]

Answer 3

(i) Let the number of vertices and edges in a simple graph G be n and m respectively and label the vertices with the numbers 1, 2, ..., n.
Let the degree at each vertex be denoted by $d_1, d_2, ..., d_n$.
Consider each edge in turn.
Each has two ends that terminate at two distinct vertices.
Thus the total degree count $\sum_1^n d_i$ is increased by 2 by each edge.
As there are m edges altogether the desired result follows, that

$$\sum_1^n d_i = 2m \qquad \square$$

[4 marks]

(ii) A graph can be used to represent a gathering of people shaking hands. Each person at the gathering is represented by a vertex. An edge between two vertices indicates those two people have shaken hands. The degree of a vertex then denotes the total number of distinct people that the associated person has shaken hands with. Of course, one handshake (one edge) involves two people (two vertices) and so the total number of people who have shaken hands is twice the number of distinct handshakes that have taken place.

[2 marks]

(iii)

Lemma 1.1 : The Even Number of Odds
Every graph has an even number of odd-degree vertices.

Any graph can be considered to be constructed from p vertices of even degree and q vertices of odd degree. Label the vertices of the graph such that the even degree vertices are $d_1, d_2, ..., d_p$

and the odd degree vertices are $d_{p+1}, d_{p+2}, ..., d_{p+q}$

The total degree count is then given by, $\sum_1^p d_i + \sum_{p+1}^{p+q} d_i$

Regardless of whether p is odd or even, the sum of all the even degree vertices will be even. This is because both an odd number of even numbers and an even number of even numbers is even.
By way of deriving a contradiction, suppose that a graph has an odd number of vertices, q, of odd degree. Now, an odd number of odd numbers is odd. So we have a total degree count that is the sum of an even and an odd number which is odd. However, from the handshake lemma we know that the total degree count must be even.
This contradiction shows that the assumption that there can be an odd number of vertices, q, of odd degree is false. Thus we deduce that the number of vertices, q, of odd degree must be even. $\square$

[5 marks]

Answer 4

(i) From the handshaking lemma we can determine m the number of edges,

$$\sum_{1}^{20} d_i = 2m$$

$$20 \times 3 = 2m \qquad \text{As each of the 20 vertices is of degree 3}$$

$$m = 30 \text{ edges}$$

Now, from Euler's Polyhedral Formula, the number of faces is found,

$$n - m + f = 2$$

$$20 - 30 + f = 2$$

$$f = 12 \quad \text{faces}$$

[4 marks]

(ii) Generalising the part (i) answer,

$$np = 2m \iff m = \frac{np}{2}$$

$$n - \frac{np}{2} + f = 2 \iff f = 2 - n + \frac{np}{2}$$

This shows that either the regularity or the number of vertices must be even as $n, m, f, p \in \mathbb{Z}^+$ □

[4 marks]

Answer 5

Lemma 1.2 : Edge Count of a Complete Graph (A Clique)

The complete graph K_n has exactly $\dfrac{n(n-1)}{2}$ edges

Proof (by induction)

Let $m(n) = \dfrac{n(n-1)}{2}, \quad n \in \mathbb{Z}^+$

where n is the number of vertices of K_n and $m(n)$ the corresponding number of edges.

When $n = 1, \quad m(1) = \dfrac{1(1-1)}{2}$

$= 0$ which is clearly true.

(The complete graph with one vertex, K_1, has no edges)

Assume that when $n = k, \; m(k) = \dfrac{k(k-1)}{2}$

If one additional vertex is added, it must connect to the k existing vertices.

In consequence, $m(k+1) = m(k) + k$

$$= \frac{k(k-1)}{2} + k$$

$$= \frac{k^2 - k}{2} + \frac{2k}{2}$$

$$= \frac{k^2 + k}{2}$$

$$= \frac{(k+1)k}{2}$$

$$= \frac{(k+1)((k+1)-1)}{2}$$

Which is precisely the assumed formula for $m(k)$ with k replaced with $k+1$

Therefore, if $m(n)$ is the number of edges when $n = k$,

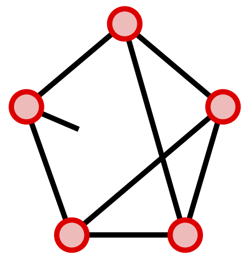
then $m(n)$ is also the number of edges when $n = k+1$

As $m(n)$ is the number of edges when $n = 1$, $m(n)$ is also the number of edges for all $n \in \mathbb{Z}^+$ by mathematical induction. $\square$

[6 marks]

Answer 6

(i) On trying to draw a 3-regular graph on 5 vertices, it's immediately apparent that it can't exist;



From the Handshaking Lemma (See question 3(i)) we know that the sum of all vertex degrees is twice the sum of the edges. We are trying to create graph that will have a vertex sum of 5×3 but this is twice 7.5 edges and so can't exist. In the figure you can literally see the 7.5 edges where the 0.5 edge has "nowhere to go".
Alternatively, invoke the result from question 3(iii) which stated that the number of vertices of odd degree must be even. We are trying to create a graph where the number of vertices of odd degree is odd.

[3 marks]

(ii) The above reasoning will apply to a r-regular graph on n vertices where the degree sum is given by rn and the number of edges by half of that. If r and n are both odd then their product is also odd and so not divisible by 2 which indicates that there will then be an edge with "nowhere to go".

That is, $m = \dfrac{rn}{2}$ is not an integer number of edges when r and n are odd.

[3 marks]

Answer 7

Lemma 1.3 : Vertex Pair Of Equal Degree
In any finite simple graph with at least two vertices, there must be at least two vertices which have the same degree.

Proof (*By contradiction*)
First a reminder that in a *simple* graph there is, at most, one edge between any vertex pair. In a graph with n vertices, a vertex can, at most, connect to all of the other $(n - 1)$ vertices. Suppose that there exists a graph with n vertices that have the different degrees. Counting down from the maximum degree of $(n - 1)$ these must be given by,

$$\{ d_1, d_2, d_3, \ldots, d_n \} = \{ 0, 1, 2, \ldots, n - 1 \}$$

However, this requires that the graph have a vertex with degree $(n - 1)$ and another of degree 0 which is a contradiction because the vertex of degree $(n - 1)$ has to connect to all other vertices. Thus there must be at least two vertices of the same degree. $\square$

[4 marks]

Answer 8

It is the wording of this question that is most likely to cause difficulty !

"Let G be a graph with nine vertices such that each vertex is of either degree 5 or of degree 6. Show that G has at least six vertices of degree 5, or at least five vertices of degree 6".

By way of understanding the question let's initially take an unsophisticated approach and say that the nine vertices could divide between those of degree 5 and those of degree 6 as follows,

$$
\begin{array}{c}
9 \times 5 + 0 \times 6 \\
8 \times 5 + 1 \times 6 \\
7 \times 5 + 2 \times 6 \\
6 \times 5 + 3 \times 6 \\
5 \times 5 + 4 \times 6 \\
4 \times 5 + 5 \times 6 \\
3 \times 5 + 6 \times 6 \\
2 \times 5 + 7 \times 6 \\
1 \times 5 + 8 \times 6 \\
0 \times 5 + 9 \times 6
\end{array}
$$

The top 4 rows of this table are the "at least six vertices of degree 5" and the bottom 5 rows are the "at least five vertices of degree 6".

So the only row that is not covered and for which a reason needs to be given for its removal is the 5 vertices of degree 5 and 4 vertices of degree 6.

This is readily done using the fact from question 3(iii) that the number of vertices in a graph of odd degree must be even.

After applying this criteria the table becomes,

$$
\begin{array}{c}
8 \times 5 + 1 \times 6 \\
6 \times 5 + 3 \times 6 \\
4 \times 5 + 5 \times 6 \\
2 \times 5 + 7 \times 5 \\
0 \times 5 + 9 \times 6
\end{array}
$$

and each row does now indeed satisfy the claim in the question. □

[4 marks]

Answer 9

In order to maximise the number of possible edges, we need to minimise the amount of disconnectedness. This will be achieved by having a graph that is only in two separated pieces, each of those pieces being maximally connected. Let k be the number of vertices in the first such piece and $(n-k)$ be the number of vertices in the second piece (such that the sum of all the vertices in the two pieces is n). From question 5 it is known that the complete graph K_n has exactly $\frac{n(n-1)}{2}$ edges. So the first piece has $\frac{k(k-1)}{2}$ edges and the second piece has $\frac{(n-k)(n-k-1)}{2}$ edges. In total the number of edges in given by,

$$\begin{aligned} E(k) &= \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\ &= \frac{k^2 - k + n^2 - nk - n - nk + k^2 + k}{2} \\ &= \frac{2k^2 - 2nk - n + n^2}{2} \end{aligned}$$

For any given graph, n is a fixed constant, and it is k that varies.

In fact, $1 \leqslant k \leqslant n-1$

$E'(k) = 2k - n$ and this gives that $E(k)$ is a minimum when $k = \frac{n}{2}$

In other words, the least number of edges is obtained by having as close to half of the vertices in each piece of the disconnected half.

The maximum number of edges, which is what we are after, will thus occur at the extremes of the inequality for k (as it's a right way up quadratic, $\cup$).

When $k = 1$,

$$\begin{aligned} E(1) &= \frac{2 - 2n - n + n^2}{2} \\ &= \frac{(n-2)(n-1)}{2} \end{aligned}$$

When $k = n-1$,

$$\begin{aligned} E(n-1) &= \frac{2(n-1)^2 - 2n(n-1) - n + n^2}{2} \\ &= \frac{2n^2 - 4n + 2 - 2n^2 + 2n - n + n^2}{2} \\ &= \frac{n^2 - 3n + 2}{2} \\ &= \frac{(n-2)(n-1)}{2} \end{aligned}$$

Thus, the maximum number of edges in a disconnected graph occurs when there is a single isolated vertex, the remaining vertices forming a complete graph with the number of edges given by $K_{n-1} = \frac{(n-2)(n-1)}{2}$ □

[6 marks]

Answer 10

(i) This can be proven using induction.

Let $s(x)$ denote the number of subsets that a set S of x elements can have.

The result to be proven is that $s(x) = 2^x$

As the base case suppose we have a set of one element.
There are two possible subsets, either the empty set, or the set with the element in it.

Thus $s(1) = 2^1 = 2$ is established as a basis for the induction.

Assume that $s(k) = 2^k$ for some positive integer value of k.
Now consider enlarging the number of elements of set S by one.

For each of the subsets of $s(k)$ a pair of subset will be counted by

$s(k + 1)$ one from adding the new element and one from not adding it.

In consequence the count for $s(k + 1)$ will be twice that of $s(k)$.

$$\begin{aligned} s(k+1) &= 2 \times s(k) \\ &= 2 \times 2^k \qquad \text{by assumption} \\ &= 2^{k+1} \end{aligned}$$

which is precisely the assumed formula for $s(k)$ with k replaced with $k + 1$

Therefore, if $s(n)$ is the number of subsets when $n = k$,

then $s(n)$ is also the number of subsets when $n = k + 1$

As $s(n)$ is the number of subsets when $n = 1$, $s(n)$ is also the number of subsets for all $n \in \mathbb{Z}^+$ by mathematical induction.

[3 marks]

(ii) Let M_{MAX} be the maximum possible number of edges, and label these.
Between any pair of vertices there is either an edge or there is not.
Determining the number of possible graphs now corresponds to determining the number of possible subsets for a set of x elements which, from part (i) is given by 2^x. That is $2^{M_{MAX}}$.
The graph on n vertices with the maximum possible number of edges is the complete graph K_n. From question 5,

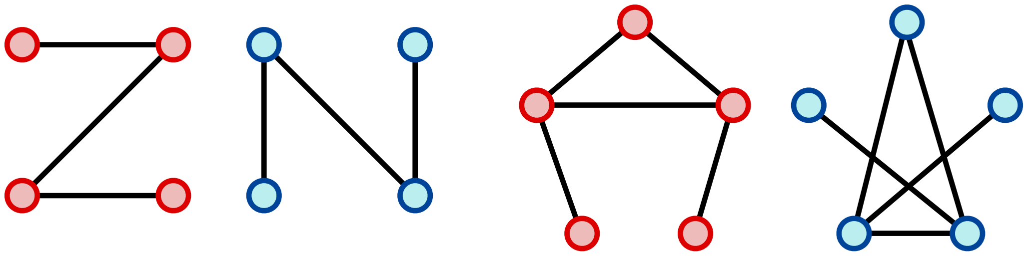
"the complete graph K_n has exactly $\dfrac{n(n-1)}{2}$ edges"

Thus there are exactly $2^{\frac{1}{2}n(n-1)}$ labelled simple graphs on n vertices. $\square$

[3 marks]

Answer 11

(i)



(1, 1, 2, 2) (1, 1, 2, 2) (1, 1, 2, 3, 3) (1, 1, 2, 3, 3)

[2 marks]

(ii) On a graph of n vertices, the maximum degree at any vertex is $n - 1$ which occurs when that vertex has an edge going to each of the other $n - 1$ vertices. In section 1.6 it was observed that "the graph sum $G + \overline{G}$ on graph of degree n is the complete graph K_n".

So, if the degree of vertex i is d_i and the degree of its complement $\overline{d_i}$

then $d_i + \overline{d_i} = n - 1$

Also, it must be the case that,

$$\begin{aligned} \overline{d_1} &= d_n, \\ \overline{d_2} &= d_{n-1} \\ \ldots &= \ldots \\ \overline{d_i} &= d_{n-(i-1)} \end{aligned}$$

That is, $\overline{d_i} = d_{n+1-i}$

Combining the two results gives that $d_i + d_{n+1-i} = n - 1$

[2 marks]

(iii) From question 5,

$$\text{“}K_n \text{ has exactly } \frac{n(n-1)}{2} \text{ edges”}$$

is obtained the fact that the complete graph K_5 will have 10 edges.
It is then deduced that a self-complementary graph on n vertices will have half of this, which for $n = 5$ is 5 edges.
The handshaking lemma then states that the sum of degrees will be double this, which in this case is 10.
The part (ii) symmetry result implies that the degree sequence must be,

$$(d_1, d_2, d_3, \ 4 - d_1, \ 4 - d_2)$$

In combination we have that,

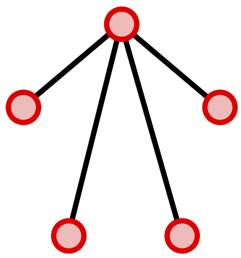
$$d_1 + d_2 + d_3 + 4 - d_1 + 4 - d_2 = 10$$

$$\text{from which we get that } d_3 = 2$$

The possibilities are now,

case	d_1	+	d_2	+	d_3	+	d_4	+	d_5
1	0	+	0	+	2	+	4	+	4
2	0	+	1	+	2	+	3	+	4
3	0	+	2	+	2	+	2	+	4
4	1	+	1	+	2	+	3	+	3
5	1	+	2	+	2	+	2	+	3
6	2	+	2	+	2	+	2	+	2

This list can be pruned further,
Cases 1, 2 and 3 require that we have a graph on 5 vertices that has a vertex of degree 4 and one of degree 0 which is clearly not possible as illustrated by the diagram below.



Out of interest, readers may like to note that Case 4 is satisfied by the graph on 5 vertices from part (i) which is the only graph (up to isomorphism) to do so. Case 6 is satisfied by C_5 as noted is section 1.6 which, again, is the only graph (up to isomorphism) to do so.
Case 5 yields no self-complementary graphs.

[4 marks]

(iv) The idea here is to generalising slightly the approach taken in part (iii) starting again will the result from question 5 that

$$m(K_n) = \frac{n(n-1)}{2}$$

Again it is deduced that a self-complementary graph on n vertices, G_n, will have half that number of edges,

$$m(G_n) = \frac{n(n-1)}{4}$$

As n and $(n-1)$ represent two consecutive integers they cannot both be even. Furthermore, $m(G_n)$ must be an integer and so either n is divisible by 4 or $(n-1)$ is divisible by 4. In other words the number of vertices of a self-complementary graph, $n(G_n)$, (or just n) has the form $4k$ or $4k + 1$ for $k \in \mathbb{Z}^+$ which is the result requested. □

[5 marks]

(v) Case 1 : For a graph of order $4k$, from part (iv),

$$\frac{4k(4k-1)}{4} = k(4k-1) \text{ edges}$$

From the handshaking lemma, the sum of the degrees of all vertices will be $2k(4k-1)$ but this cannot be shared equally among $4k$ vertices. That is,

$$\frac{2k(4k-1)}{4k} = \frac{4k-1}{2} \text{ with } k \in \mathbb{Z} \text{ is not an integer.}$$

Thus, $n \neq 0 \pmod 4$

Case 2 : For a graph of order $4k + 1$, from part (iv),

$$\frac{(4k+1)(4k+1-1)}{4} = k(4k+1) \text{ edges}$$

From the handshaking lemma, the sum of the degrees of all vertices will be $2k(4k+1)$ which can be shared equally among the $4k + 1$ vertices.

Thus, $n \equiv 1 \pmod 4$

Overall we conclude that if G is a self-complementary regular graph on n vertices then $n \equiv 1 \pmod 4$ □

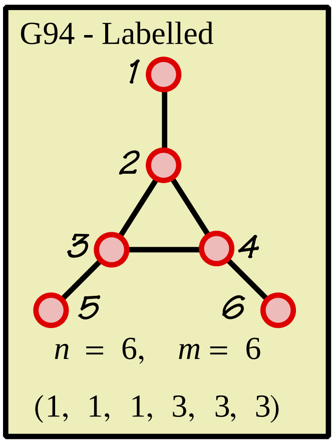
[5 marks]

Lecture 2

Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

2.1 The Adjacency Matrix

By definition, the vertices v and w of a graph are *adjacent* vertices if they are joined by an edge, e. If G is a graph with n vertices (labelled 1, 2, 3, ..., n) then the *adjacency matrix* $\mathbf{A}(G)$ of G is the $n \times n$ square matrix in which the entry a_{ij} is the number of edges joining the vertices i and j. For a graph that is *simple* the entry can only be 0 or 1. Below, as an example, is the graph G94 which has been labelled and next to it is given its adjacency matrix. The top row of this adjacency matrix shows that the vertex labelled *1* is connected only to the vertex labelled *2*, the second row shows that the vertex labelled *2* has a direct connection to the vertices labelled *1*, *3* and *4*, and subsequent rows show how the remaining vertices connect.



$$\mathbf{A}(\mathrm{G94}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In general the adjacency matrix of a simple graph will be symmetric and have a leading diagonal of all zeros. Interest in adjacency matrices centres around identifying properties of graphs that are captured by them. For example, the trace of a square matrix is the sum of its diagonal entries and denoted by $tr(\mathbf{A})$. It turns out that $tr\left(\mathbf{A}^2\right)$ gives twice the number of edges of the associated graph and $tr\left(\mathbf{A}^3\right)$ gives six times the number of triangles, as illustrated below.

$$\mathbf{A}^2(\mathrm{G94}) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow tr\left(\mathbf{A}^2(\mathrm{G94})\right) = 12 \quad \therefore 6 \textit{ edges}$$

$$\mathbf{A}^3(\mathrm{G94}) = \begin{pmatrix} 0 & 3 & 1 & 1 & 1 & 1 \\ 3 & 2 & 5 & 5 & 1 & 1 \\ 1 & 5 & 2 & 5 & 3 & 1 \\ 1 & 5 & 5 & 2 & 1 & 3 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 & 1 & 0 \end{pmatrix} \Rightarrow tr\left(\mathbf{A}^3(\mathrm{G94})\right) = 6 \quad \therefore 1 \textit{ triangle}$$

Performing calculations and manipulations on large matrices is tedious by hand and more reliably done using computer software.

Let $\phi(X, x)$ denote the characteristic polynomial of $\mathbf{A}(x)$.

For the adjacency matrix $\mathbf{A}$(G94) software gives its characteristic equation as,

$$\phi(X, x) = \left(x^2 - 2x - 1\right)\left(x^2 + x - 1\right)^2$$

The spectrum of a matrix is the list of its eigenvalues together with their multiplicities. For $\mathbf{A}$(G94) the spectrum is,

$$\left\{1 \pm \sqrt{2}, \frac{-1 \pm \sqrt{5}}{2}^{(2)}\right\}$$

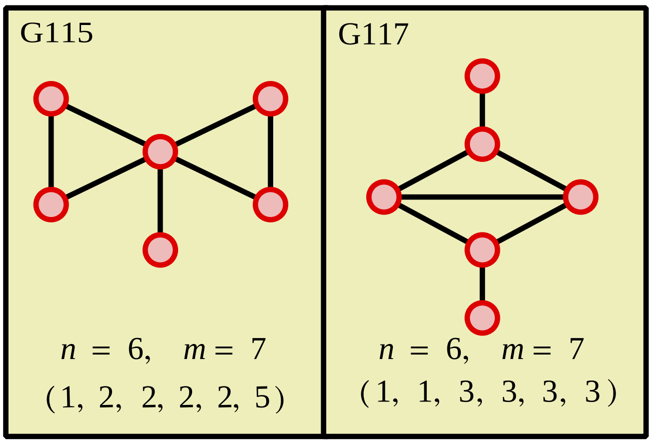
where the superscripts give the multiplicities that are greater than one.

Let $\alpha = 1 \pm \sqrt{2}$, and $\beta = \dfrac{-1 \pm \sqrt{5}}{2}$

The six eigenvectors of $\mathbf{A}$(G94) are then,

$$v(\lambda = \alpha) = \begin{pmatrix} 1 \\ \alpha \\ \alpha \\ \alpha \\ 1 \\ 1 \end{pmatrix} \qquad v(\lambda = \beta) = \begin{pmatrix} -1 \\ -\beta \\ 0 \\ \beta \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -\beta \\ \beta \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Of ongoing interest is determining if two graphs are isomorphic from the mathematics associated with them. If two graphs, G and H are isomorphic then, although they have different adjacency matrices $\mathbf{A}(G)$ and $\mathbf{A}(H)$, they will have the same characteristic equation and spectrum. However, this cannot be used the other way round; cospectral graphs are not isomorphic, yet have the same characteristic equation and spectrum.



G115 and G117 provide an example of cospectral graphs. They are clearly not isomorphic as G115 has one vertex of degree 1 (a leaf) whereas G117 has two. Yet they both have the same characteristic equation.

$$\phi(\text{G115}, x) = \phi(\text{G117}, x) = \left(x - 1\right)\left(x + 1\right)^2\left(x^3 - x^2 - 5x + 1\right)$$

An objective of this lecture is to show that, in spite of the cospectral set back, it is possible to determine if two graphs are isomorphic from their adjacency matrices. However, to do so requires the prior development of a few ideas and it is to these we now attend.

2.2 Transposed Matrices

Here is a brief reminder of what the transpose of a matrix is.

The Transpose of an $n \times n$ Matrix

Given the matrix $\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ the transpose of matrix **M**

is denoted $\mathbf{M}^{\mathrm{T}}$ and is formed by an interchange of rows and columns.

Thus, $$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

2.3 Symmetric Matrices

A matrix, **M**, is symmetric if $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$. Such matrices are readily recognised for their elements are symmetric with respect to the leading diagonal. The adjacency matrix of a graph is symmetric as are powers of that matrix which means that the properties of such matrices will be of importance.

2.4 Permutation Matrices

A permutation matrix, **P**, is an $n \times n$ square matrix such that each row and each column contains a single element equal to 1, the remaining elements being 0.
Consider the following calculation which involves a permutation matrix,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} d \\ b \\ a \\ e \\ c \end{pmatrix}$$

This permutation matrix has permutated the letters a, b, c, d, e as shown below,

$$\begin{pmatrix} a & b & c & d & e \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ d & b & a & e & c \end{pmatrix}$$

In cycle notation this permutation could be written ($adec$)(b) or just ($adec$).

A permutation matrix can permutate an entire matrix in two different ways as the following two calculations illustrate,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & f & k & p & u \\ b & g & l & q & v \\ c & h & m & r & w \\ d & i & n & s & x \\ e & j & o & t & y \end{pmatrix} = \begin{pmatrix} d & i & n & s & x \\ b & g & l & q & v \\ a & f & k & p & u \\ e & j & o & t & y \\ c & h & m & r & w \end{pmatrix}$$

$$\begin{pmatrix} a & f & k & p & u \\ b & g & l & q & v \\ c & h & m & r & w \\ d & i & n & s & x \\ e & j & o & t & y \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} p & f & a & u & k \\ q & g & b & v & l \\ r & h & c & w & m \\ s & i & d & x & n \\ t & j & e & y & o \end{pmatrix}$$

In the first calculation it is the rows of the lettered matrix that have been permutated. Notice that in the second of these calculations the transpose of the permutation matrix has been used. In this calculation it is the columns of the lettered matrix that have been permutated.

All of this illustrates the next theorem.

Theorem 2.1 : Permutating Rows and Columns

Given a square matrix, **S**, and a permutation matrix, **P**, multiplying by **P** on the left permutates the rows of **S**, whilst multiplying by $\mathbf{P}^{\mathrm{T}}$ on the right permutates the columns.

PS permutates rows of **S**

$\mathbf{SP}^{\mathrm{T}}$ permutates columns of **S**

In general there are $n!$ permutation matrices of size $n \times n$. Of the six 3×3 permutation matrices three are elementary permutation matrices that swap just two rows or two columns. Here are those six matrices. Those that are elementary are highlighted in red, and the (left multiplying) permutating effect they would have on the column vector $(a,\ b,\ c)^{\mathrm{T}}$ is given immediately underneath each.

$$\underset{(a)(b)(c)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}},\ \underset{(bc)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}},\ \underset{(ac)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}},\ \underset{(ab)}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}},\ \underset{(acb)}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}},\ \underset{(abc)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

In general, of the $n!$ permutation matrices of dimension $n \times n$ the number that are elementary is given by the triangular number T_{n-1} where,

$$T_n = \frac{n(n+1)}{2}$$

Clearly, any permutation matrix raised to a sufficient power will yield the identity matrix, **I**. The following demonstrates this fact for each of the six 3 × 3 permutation matrices;

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}^2 = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}^2 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}^2 = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}^3 = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}^3$$

$$(a)\,(b)\,(c) \quad = \quad (bc)^2 \quad = \quad (ac)^2 \quad = \quad (ab)^2 \quad = \quad (acb)^3 \quad = \quad (abc)^3$$

In general, a non-elementary permutation matrix can be decomposed into a product of elementary permutation matrices. Again, this is a fact that can be demonstrated for the 3 × 3s, although there is a catch, shortly to be explained;

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}^2 = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}^2 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}^2$$

$$(a)\,(b)\,(c) \quad = \quad (bc)^2 \quad = \quad (ac)^2 \quad = \quad (ab)^2$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$(abc) \quad = \quad (ac) \quad (ab)$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$(acb) \quad = \quad (ab) \quad (ac)$$

The astute reader looking at the above three matrix equations may be wondering why the matrix $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ can be associated with different permutations. This is a much glossed over issue that an internet search will do little to explain. When this matrix acts on $(a, b, c)^{\mathrm{T}}$ it represents the permutation (bc). However when it acts upon $(b, a, c)^{\mathrm{T}}$ it represents (ac) and when it acts upon $(c, b, a)^{\mathrm{T}}$ it represents the permutation (ab). Marrying up a matrix with the permutation it represents is not as straight forward as one might initially have expected; it depends upon how preceding matrices have permutated the rows (or columns).

Permutations and their manipulation in cycle notation are covered in the Number Wonder undergraduate lectures, Group Theory II.
https://www.NumberWonder.co.uk/Pages/Page9110.html

2.5 Orthogonality

A matrix, **Q**, is described as being orthogonal if it is a real square matrix and has the property that $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ where $\mathbf{Q}^{\mathrm{T}}$ is the transpose of **Q** and **I** is the identity matrix. This immediately leads to an equivalent characterization of orthogonality; **Q** is orthogonal if its transpose is equal to its inverse, if $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}$

Definition : An Orthogonal Matrix
A real matrix is orthogonal iff it is invertible and its inverse is its transpose.

The interest in orthogonality stems from the fact that permutation matrices have this property. For example, here the permutation introduced at the start of section 2.4 is multiplied by its transpose and the result, as claimed, is indeed the 5 × 5 identity matrix,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 2.2 : Permutation Matrices are Orthogonal
The product of a permutation matrix and its transpose gives the identity matrix.

That is, $$\mathbf{P}\,\mathbf{P}^{\mathrm{T}} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$$

If follows that permutation matrices are orthogonal.

Proof

Clearly the $n \times n$ identity matrix, **I**, is orthogonal for all positive integer values of n. If any two rows in **I** or any two columns in **I** are swapped the result is an elementary permutation matrix which retains the property of being orthogonal because it is still symmetric and still coincides with its inverse. This proves the theorem in the case of the elementary permutation matrices.

Any non-elementary permutation matrix, **P**, can be decomposed into a product of elementary permutation matrices, $\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_k$ and we now argue as follows;

$$\mathbf{P}^{-1} = \left(\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_k\right)^{-1} = \mathbf{P}_k^{-1} \ldots \mathbf{P}_2^{-1}\,\mathbf{P}_1^{-1} = \mathbf{P}_k^{\mathrm{T}} \ldots \mathbf{P}_2^{\mathrm{T}}\,\mathbf{P}_1^{\mathrm{T}} = \left(\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_k\right)^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}}$$

which completes the proof.

□

2.6 Isomorphism

For two graphs to be isomorphic there are many properties that must be common. They must have the same number of vertices, or edges, or spectrum, for example. However, non isomorphic graphs can have the same number of edges, for example, and it was shown previously that non isomorphic graphs can even have the same spectrum (cospectral graphs). Determining if two graphs are isomorphic or not can be a frustrating business. As all of the structure of a graph is captured by its adjacency matrix it is in principle possible to determine if two graphs are isomorphic. Write down the adjacency matrix of each, and then search for a permutation matrix for which Theorem 2.3, stated next, holds.

Theorem 2.3 : Isomorphism via Adjacency Matrices
Let G and H be graphs on the same vertex set and with adjacency matrices $\mathbf{A}(G)$ and $\mathbf{A}(H)$ respectively. Then G and H are isomorphic if and only if there is a permutation matrix $\mathbf{P}$ such that,

$$\mathbf{P}^{T}\mathbf{A}(G)\,\mathbf{P} = \mathbf{A}(H)$$

Being able to state Theorem 2.3 and to have developed the mathematics to understand what it is saying has been the goal of this lecture. However, in some respects it is a damp squib. This is because in practice, theorem 2.3 is of limited use in the general case; for a graph with n vertices, there are $n!$ candidates to be the sought after permutation matrix.

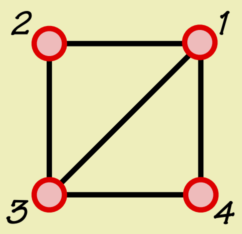
2.7 Exercises

Marks Available : 80

Question 1

(i) Write down the adjacency matrix $\mathbf{A}$(G17) for G17, shown below,

G17 - Labelled



$n = 4, \quad m = 5$

(2, 2, 3, 3)

[2 marks]

(ii) By hand, write down the matrix $\mathbf{A}^2$(G17) which will give the number of walks of length 2 between the various vertices.

[2 marks]

(iii) Verify from the graph of G17 that there are three walks of length 2 between vertex 1 and itself.

[1 mark]

Question 2

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$ be the adjacency matrix of a labelled graph G

where the entry a_{ij} is 1 if there is an edge between vertex v_i and v_j, and 0 otherwise.

(i) Write down an expression for the top left entry of $\mathbf{A}^2$

[1 mark]

(ii) Explain why this counts the number of walks of length 2 between vertex 1 and itself.

[2 marks]

(iii) Building on your part (ii) answer, explain why $tr\left(\mathbf{A}^2\right)$ gives twice the number of edges of the associated graph.

[2 marks]

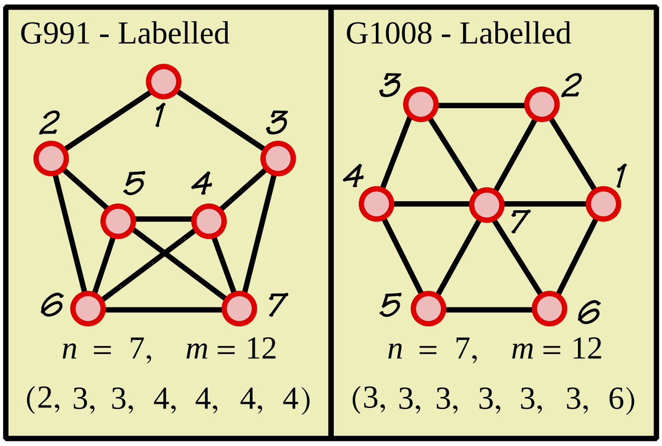
(iv) A graph, H, has adjacency matrix $\mathbf{H}$ such that $\mathbf{H}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

Draw the graph of H.

[2 marks]

Question 3

The graphs G991 and G1008, shown below, are clearly not isomorphic as they have different degree sequences.



(i) Write down the adjacency matrix for each graph.

[2 marks]

(ii) Use software to find the characteristic polynomials for G991 and G1008 and hence deduce that these two graphs are cospectral.

[4 marks]

(iii) Find the characteristic polynomial for the square of each adjacency matrix and comment.

[4 marks]

Question 4

Theorem 2.4 : Counting walks between vertices
Given a simple graph G with adjacency matrix $\mathbf{A}$, raising $\mathbf{A}$ to the power n gives a matrix where the entry a_{ij} gives the number of walks of length n between the vertices v_i and v_j

---

Write out a proof by induction for Theorem 2.4

[6 marks]

Question 5

For a simple graph G with adjacency matrix $\mathbf{A}$, explain why $tr\left(\mathbf{A}^3\right)$ gives six times the number of triangles in G.
You may quote Theorem 2.4 (from question 4) as a part of your explanation.

[4 marks]

Question 6

Lemma 2.1 : Disconnected Detector
For a graph G of order n and adjacency matrix $\mathbf{A}$, calculate matrix $\mathbf{S}_n$ where,

$$\mathbf{S}_n = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots + \mathbf{A}^n$$

If there are any zeros in $\mathbf{S}_n$ then the graph is not connected.

Give a short proof of Lemma 2.1
You may quote Theorem 2.4 (from question 4)as a part of your explanation.

[3 marks]

Question 7

Lemma 2.2 Shortest Path Between a Vertex Pair

For a graph G of order n and adjacency matrix $\mathbf{A}$, calculate matrix $\mathbf{S}_k$ where,

$$\mathbf{S}_k = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots + \mathbf{A}^k, \quad k \leqslant n$$

The entry in row i and column j of matrix $\mathbf{S}_k$ tallies the number of ways to get from vertex v_i to vertex v_j in k steps or less. (A step is the traversal of an edge). To find the shortest number of steps between v_i and v_j begin to calculate the partial sums $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, ..., $\mathbf{S}_n$. Then, the first value of k for which the entry in row i and column j of matrix $\mathbf{S}_k$ is non-zero is the shortest number of steps.

A graph, G, has adjacency matrix, $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

(i) As necessary, use software to write down $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$

[4 marks]

(ii) Use your part (i) answer to write out the Shimbel Matrix, $\mathbf{M}$, for G where the entry row i and column j of matrix $\mathbf{M}$ is the least number of steps between the vertices v_i and v_j

[3 marks]

(iii) The diameter of a graph is the shortest path between the most distant distant minimum vertices. This is the largest value in the Shimbel Matrix. State the diameter of G.

[1 mark]

Question 8

A “simple” 4-cycle is a closed walk around four distinct vertices of the form

v_a - v_b - v_c - v_d - v_a

This excludes walks of the form v_a - v_b - v_a - v_b - v_a

and v_a - v_b - v_a - v_d - v_a

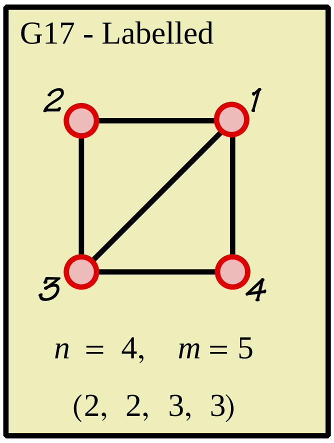
and v_a - v_b - v_c - v_b - v_a

Algorithm 2.1 : Counting Simple 4-Cycles

For a graph G with adjacency matrix **A**, the number of proper 4-cycles is,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

(i) From question 1, for the graph of G17 we know that,



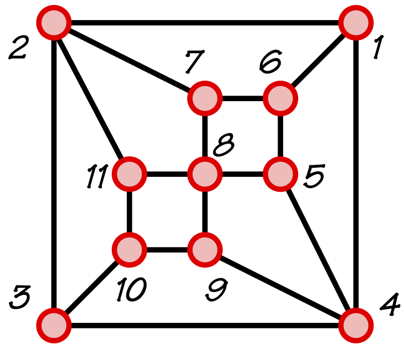
$$\mathbf{A}(\text{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^2(\text{G17}) = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Show that Algorithm 2.1 correctly finds a single simple 4-cycle in G17.

[3 marks]

(ii) Use Algorithm 2.1 to find the number of simple 4-cycles in the graph G877 which is shown below. This graph is sufficiently small so that you can see what the correct answer should be !



[4 marks]

Question 9
Using matrix methods, how many simple 4-cycles are there in the graph,

(i) K_4

[2 marks]

(ii) K_7

[3 marks]

(iii) K_n

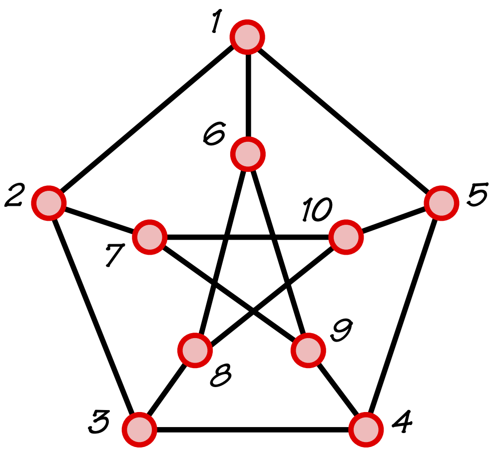
[5 marks]

Question 10

Algorithm 2.2 : Counting Simple 5-Cycles
For a graph G with adjacency matrix $\mathbf{A}$, the number of proper 5-cycles is,

$$\frac{1}{10}\left(tr\left(\mathbf{A}^5\right) - 5\sum_{i=1}^{n}\left(\left(a_{ii}^{(3)}\right)\left(a_{ii}^{(2)} - 2\right)\right) - 5\,tr\left(\mathbf{A}^3\right)\right)$$

Shown is the graph and adjacency matrix for the Petersen graph.



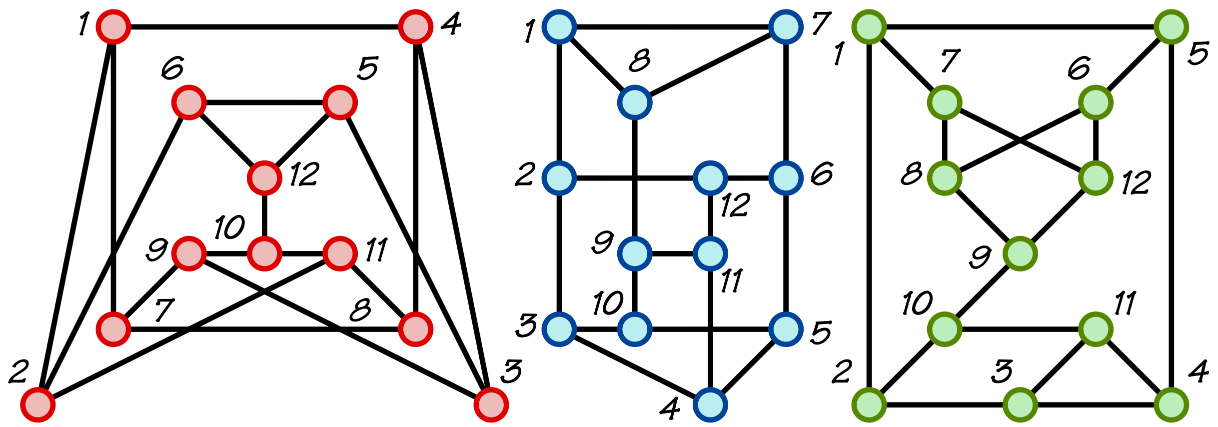
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Use Algorithm 2.1 to find the number of simple 5-cycles in the Petersen graph.

[5 marks]

Question 11

The purpose of this question is to investigate the isomorphisms (if any) between the three (labelled) graphs presented below.



(i) State the number of vertices of each graph.
Does this identify if any of the three are non-isomorphic to the others ?

[1 mark]

(ii) State the number of edges of each graph.
Does this identify if any of the three are non-isomorphic to the others ?

[1 mark]

(iii) State the degree sequence of each graph.
Does this identify if any of the three are non-isomorphic to the others ?

[1 mark]

(iv) Construct the adjacency matrix for each graph.

[3 marks]

(v) Use computer software to cube each graph's adjacency matrix.
For each of these matrices calculate $tr\left(\mathbf{A}^3\right)$.
Hence state the number of triangles in each graph.
Does this identify if any of the three are non-isomporphic to the others ?

[3 marks]

(vi) Use software to determine the characteristic equation for each graph's adjacency matrix and hence find the associated spectrum.
Does this identify if any of the three are non-isomporphic to the others ?

[6 marks]

2.8 Answers to 2.7 Exercise

Answer 1

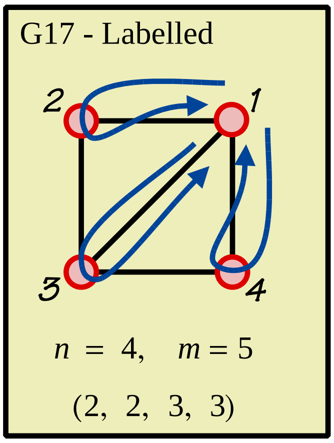
(i) $$\mathbf{A}(\mathrm{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

[2 marks]

(ii) $$\mathbf{A}^2(\mathrm{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

[2 marks]

(iii) The three walks of length 2 between vertex *1* and itself are,



[1 mark]

Answer 2

$$\mathbf{A}^2 = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

(i) $b_{11} = (a_{11}\, a_{11}) + (a_{12}\, a_{21}) + (a_{13}\, a_{31}) + \dots + (a_{1n}\, a_{n1})$

As $\mathbf{A}$ is symmetric, $a_{ij} = a_{ji}$, so $b_{11} = (a_{11})^2 + (a_{12})^2 + (a_{13})^2 + \dots + (a_{1n})^2$

[1 mark]

(ii) Now, a_{11} is always zero, and each of the squares $(a_{1k})^2$ for $2 \leqslant k \leqslant n$ will be 1 when there is an edge between vertices v_1 and v_k, 0 otherwise. Thus b_{11} gives the degree of vertex v_1 and also the number of walks of length 2 between v_1 and itself. □

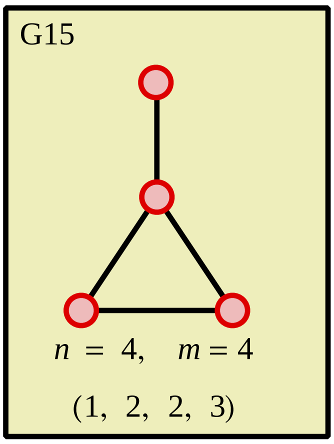
[2 marks]

(iii) $tr\left(\mathbf{A}^2\right)$ will give the sum of the degrees of all vertices in G which, by Theorem 1.2, The Handshaking Lemma, is twice the number of edges of G.

[2 marks]

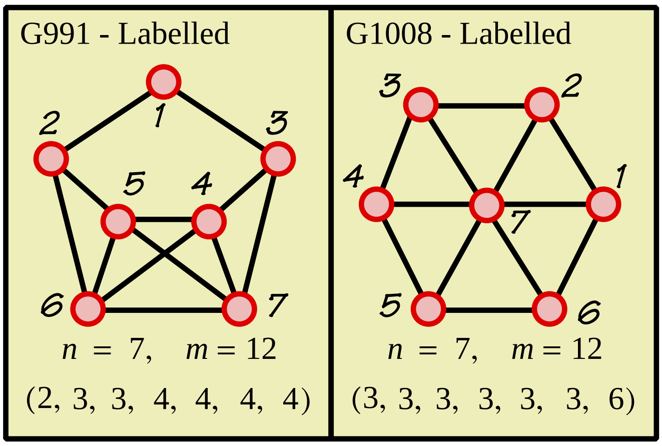
(iv) $$\mathbf{H}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

The graph H has 4 vertices and degree sequence (1, 2, 2, 3).
This is enough to identify the graph as being G15



[2 marks]

Answer 3



(i) $\mathbf{A}(\text{G991}) = \begin{pmatrix} 0&1&1&0&0&0&0\\ 1&0&0&0&1&1&0\\ 1&0&0&1&0&0&1\\ 0&0&1&0&1&1&1\\ 0&1&0&1&0&1&1\\ 0&1&0&1&1&0&1\\ 0&0&1&1&1&1&0 \end{pmatrix}$ $\mathbf{A}(\text{G1008}) = \begin{pmatrix} 0&1&0&0&0&1&1\\ 1&0&1&0&0&0&1\\ 0&1&0&1&0&0&1\\ 0&0&1&0&1&0&1\\ 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&1&1&1&1&0 \end{pmatrix}$

[2 marks]

(ii) $\phi(\text{G991}) = \phi(\text{G1008}) = (x-1)^2(x+1)^2(x+2)(x^2-2x-6)$

The two graphs are not isomorphic, yet their adjacency matrices have the same characteristic equation which, by definition, makes them cospectral.

[4 marks]

(iii) $\mathbf{A}^2(\text{G991}) = \begin{pmatrix} 2&0&0&1&1&1&1\\ 0&3&1&2&1&1&2\\ 0&1&3&1&2&2&1\\ 1&2&1&4&2&2&3\\ 1&1&2&2&4&3&2\\ 1&1&2&2&3&4&2\\ 1&2&1&3&2&2&4 \end{pmatrix}$ $\mathbf{A}^2(\text{G1008}) = \begin{pmatrix} 3&1&2&1&2&1&2\\ 1&3&1&2&1&2&2\\ 2&1&3&1&2&1&2\\ 1&2&1&3&1&2&2\\ 2&1&2&1&3&1&2\\ 1&2&1&2&1&3&2\\ 2&2&2&2&2&2&6 \end{pmatrix}$

$\phi_2(\text{G991}) = \phi_2(\text{G1008}) = (x-4)(x-1)^4(x^2-16x+36)$

Comment : The hope that the characteristic polynomials of the squares of the adjacency matrices might distinguish between the cospectral graphs is fundamentally flawed because,

"The matrix $\mathbf{A}^n$ has eigenvalue λ^n where λ is an eigenvalue of $\mathbf{A}$"

This statement may be proven using induction.
For a proof see Number Wonder's Matrix Algebra, Lecture 1, Question 5
https://www.NumberWonder.co.uk/Pages/Page9116.html

[4 marks]

Answer 4

Theorem 2.4 : Counting walks between vertices
Given a simple graph G with adjacency matrix $\mathbf{A}$, raising $\mathbf{A}$ to a positive integer power n gives a matrix where the entry a_{ij} gives the number of walks of length n between the vertices v_i and v_j

Proof

To establish a basis for a proof by induction let $n = 1$ giving $\mathbf{A}^1 = \mathbf{A}$ which is the adjacency matrix for G in which entry $a_{ij}^{(1)}$ counts the number of walks of length 1 between v_i and v_j. As G is simple this count is either 1 if there is an edge between v_i and v_j or 0 if there is no edge.
The induction hypothesis is to assume true that when $n = k$ the number of walks of length k between v_i and v_j is the entry $a_{ij}^{(k)}$ in the matrix $\mathbf{A}^k$.
We can express a walk of length $k + 1$ between v_i and v_j of a walk of length k between v_i and v_u followed by a walk of length 1 from v_u to v_j.
In consequence, the number of walks of length $k + 1$ between v_i and v_j is the sum of all walks of length k from v_i to v_u multiplied by the number of ways to walk in one step from v_u to v_j. which is given by,

$$\sum_{r=1}^{n} a_{ir}^{(k)}\, a_{rj}$$

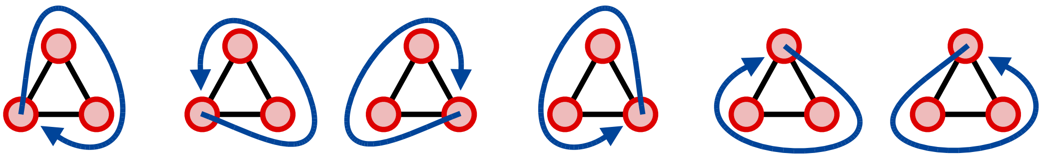
By the definition of matrix multiplication, this is the entry $a_{ij}^{(k+1)}$ in $\mathbf{A}^{k+1}$
Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction. □

[6 marks]

Answer 5

From Theorem 2.4 we know that, given a simple graph G with adjacency matrix $\mathbf{A}$, the elements on the diagonal of $\mathbf{A}^3$ (which are of the form $a_{ii}^{(3)}$) will be the walks of length 3 that start and finish at the same vertex. The only way that a walk of 3 steps can start and finish at the same vertex is if it is triangular. Let G be of order n.

The trace of $\mathbf{A}^3$ is $\sum_{i=1}^{n} a_{ii}^{(3)}$ which will be the sum of all triangular walks in G but with each counted six times as shown below.



Hence $tr\left(\mathbf{A}^3\right)$ gives six times the number of triangles in G. □

[4 marks]

Answer 6

Lemma 2.1 : Disconnected Detector

For a graph G of order n and adjacency matrix $\mathbf{A}$, calculate matrix $\mathbf{S}_n$ where,

$$\mathbf{S}_n = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \dots + \mathbf{A}^n$$

If there are any zeros in $\mathbf{S}_n$ then the graph is not connected.

Proof

If a graph is connected then the maximum length of a trail (a walk that does not traverse any edge more than once) is n. From theorem 2.4 we know that entries in the matrix $\mathbf{A}^k$ gives the number of walk of length k between all possible pairs of vertices in G. Thus a zero anywhere in the matrix $\mathbf{S}_n$ is telling us that between a pair of vertices in the graph there is no walk of length 1, 2, 3, ..., n. Thus there is a pair of vertices that have no way of connecting to each other.
In other words, the graph is disconnected. □

Note that there are other, more efficient, methods (especially as n becomes large) to determine if a graph is connected or not and, indeed, to determine the number of component parts.

[3 marks]

Answer 7

(i)

$$\mathbf{A} = \begin{pmatrix} 0&1&0&0&0&0&0 \\ 1&0&1&1&0&0&0 \\ 0&1&0&1&0&0&0 \\ 0&1&1&0&1&0&0 \\ 0&0&0&1&0&1&1 \\ 0&0&0&0&1&0&1 \\ 0&0&0&0&1&1&0 \end{pmatrix}, \quad \mathbf{A}^2 = \begin{pmatrix} 1&0&1&1&0&0&0 \\ 0&3&1&1&1&0&0 \\ 1&1&2&1&1&0&0 \\ 1&1&1&3&0&1&1 \\ 0&1&1&0&3&1&1 \\ 0&0&0&1&1&2&1 \\ 0&0&0&1&1&1&2 \end{pmatrix}$$

$$\mathbf{A}^3 = \begin{pmatrix} 0&3&1&1&1&0&0 \\ 3&2&4&5&1&1&1 \\ 1&4&2&4&1&1&1 \\ 1&5&4&2&5&1&1 \\ 1&1&1&5&2&4&4 \\ 0&1&1&1&4&2&3 \\ 0&1&1&1&4&3&2 \end{pmatrix}, \quad \mathbf{A}^4 = \begin{pmatrix} 3&2&4&5&1&1&1 \\ 2&12&7&7&7&2&2 \\ 4&7&8&7&6&2&2 \\ 5&7&7&14&4&6&6 \\ 1&7&6&4&13&6&6 \\ 1&2&2&6&6&7&6 \\ 1&2&2&6&6&6&7 \end{pmatrix}$$

[4 marks]

(ii) The Shimbel Matrix, **M**, is,

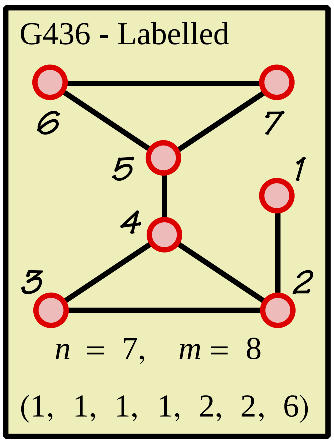
$$\mathbf{M} = \begin{pmatrix} 2&1&2&2&3&4&4 \\ 1&2&1&1&2&3&3 \\ 2&1&2&1&2&3&3 \\ 2&1&1&2&1&2&2 \\ 3&2&2&1&2&1&1 \\ 4&3&3&2&1&2&1 \\ 4&3&3&2&1&1&2 \end{pmatrix}$$

[3 marks]

(iii) Diameter is 4

[1 mark]

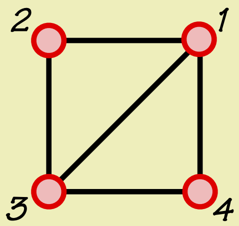
Note that another way to answer this question would be to use the adjacency matrix to draw the graph and then simply study the graph to obtain **M**.



Answer 8

(i)

G17 - Labelled



$n = 4, \quad m = 5$

$(2, \; 2, \; 3, \; 3)$

$$\mathbf{A}(G17) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^2(G17) = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A}^3(G17) = \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{pmatrix} \qquad \mathbf{A}^4(G17) = \begin{pmatrix} 15 & 9 & 14 & 9 \\ 9 & 10 & 9 & 10 \\ 14 & 9 & 15 & 9 \\ 9 & 10 & 9 & 10 \end{pmatrix}$$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(50 - 2\left(3 \times 2 + 2 \times 1 + 3 \times 2 + 2 \times 1 \right) - 10 \right)$$

$$= \frac{1}{8}\left(50 - 2 \times 16 - 10 \right)$$

$= 1$ simple 4-cycle

[3 marks]

(ii)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{A}^2 = \begin{pmatrix} 3 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 3 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 4 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 3 \end{pmatrix}$$

$2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right)$ "*twice the sum of the triangularised degrees*"

$$= 2\left(6 + 12 + 6 + 12 + 6 + 6 + 6 + 12 + 6 + 6 + 6 \right.$$

$$= 2 \times 84$$

$$= 168$$

$$tr\left(\mathbf{A}^2\right) = 36 \text{ “\textit{the sum of all degrees}”}$$

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 8 & 0 & 8 & 0 & 7 & 0 & 6 & 0 & 4 & 0 \\ 8 & 0 & 8 & 0 & 6 & 0 & 8 & 0 & 6 & 0 & 8 \\ 0 & 8 & 0 & 8 & 0 & 4 & 0 & 6 & 0 & 7 & 0 \\ 8 & 0 & 8 & 0 & 8 & 0 & 6 & 0 & 8 & 0 & 6 \\ 0 & 6 & 0 & 8 & 0 & 7 & 0 & 8 & 0 & 4 & 0 \\ 7 & 0 & 4 & 0 & 7 & 0 & 7 & 0 & 4 & 0 & 4 \\ 0 & 8 & 0 & 6 & 0 & 7 & 0 & 8 & 0 & 4 & 0 \\ 6 & 0 & 6 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \\ 0 & 6 & 0 & 8 & 0 & 4 & 0 & 8 & 0 & 7 & 0 \\ 4 & 0 & 7 & 0 & 4 & 0 & 4 & 0 & 7 & 0 & 7 \\ 0 & 8 & 0 & 6 & 0 & 4 & 0 & 8 & 0 & 7 & 0 \end{pmatrix}$$

$$\mathbf{A}^4 = \begin{pmatrix} 23 & 0 & 20 & 0 & 21 & 0 & 21 & 0 & 18 & 0 & 18 \\ 0 & 32 & 0 & 28 & 0 & 22 & 0 & 28 & 0 & 22 & 0 \\ 20 & 0 & 23 & 0 & 18 & 0 & 18 & 0 & 21 & 0 & 21 \\ 0 & 28 & 0 & 32 & 0 & 22 & 0 & 28 & 0 & 22 & 0 \\ 21 & 0 & 18 & 0 & 23 & 0 & 21 & 0 & 20 & 0 & 18 \\ 0 & 22 & 0 & 22 & 0 & 21 & 0 & 22 & 0 & 12 & 0 \\ 21 & 0 & 18 & 0 & 21 & 0 & 23 & 0 & 18 & 0 & 20 \\ 0 & 28 & 0 & 28 & 0 & 22 & 0 & 32 & 0 & 22 & 0 \\ 18 & 0 & 21 & 0 & 20 & 0 & 18 & 0 & 23 & 0 & 21 \\ 0 & 22 & 0 & 22 & 0 & 12 & 0 & 22 & 0 & 21 & 0 \\ 18 & 0 & 21 & 0 & 18 & 0 & 20 & 0 & 21 & 0 & 23 \end{pmatrix}$$

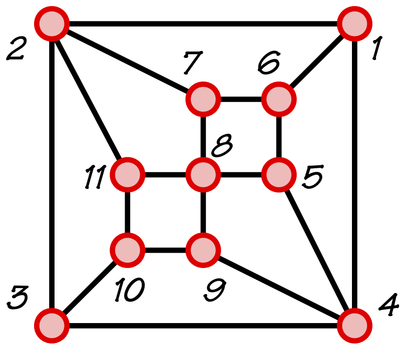
By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(276 - 168 - 36\right)$$

$$= 9 \text{ simple 4-cycles}$$

From an inspection of the graph it can be seen that this is correct.



[4 marks]

Answer 9

(i) $\mathbf{A}(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ $\quad \mathbf{A}^2(K_4) = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$

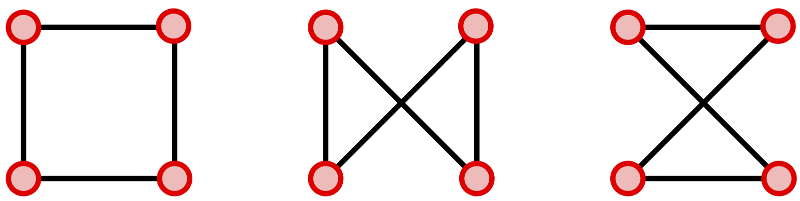
$\mathbf{A}^3(K_4) = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix}$ $\quad \mathbf{A}^4(K_4) = \begin{pmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{pmatrix}$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(84 - 2 \times 24 - 12 \right)$$

$$= 3 \text{ simple 4-cycles}$$



[2 marks]

(ii) $\mathbf{A}(K_7) = \begin{pmatrix} 0 & & & \text{\Large 1} & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & \text{\Large 1} & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$ $\quad \mathbf{A}^2(K_7) = \begin{pmatrix} 6 & & & \text{\Large 5} & & & \\ & 6 & & & & & \\ & & 6 & & & & \\ & & & 6 & & & \\ & \text{\Large 5} & & & 6 & & \\ & & & & & 6 & \\ & & & & & & 6 \end{pmatrix}$

$\mathbf{A}^3(K_7) = \begin{pmatrix} 30 & & & \text{\Large 31} & & & \\ & 30 & & & & & \\ & & 30 & & & & \\ & & & 30 & & & \\ & \text{\Large 31} & & & 30 & & \\ & & & & & 30 & \\ & & & & & & 30 \end{pmatrix}$ $\quad \mathbf{A}^4(K_7) = \begin{pmatrix} 186 & & & \text{\Large 185} & & & \\ & 186 & & & & & \\ & & 186 & & & & \\ & & & 186 & & & \\ & \text{\Large 185} & & & 186 & & \\ & & & & & 186 & \\ & & & & & & 186 \end{pmatrix}$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(1302 - 2 \times 210 - 42\right)$$

$$= 105 \text{ simple 4-cycles}$$

[3 marks]

(iii) This is about generalising parts (i) and (ii)

$$\mathbf{A}(K_n) = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \quad \mathbf{A}^2(K_n) = \begin{pmatrix} n-1 & & n-2 \\ & \ddots & \\ n-2 & & n-1 \end{pmatrix}$$

$$\mathbf{A}^3(K_n) = \begin{pmatrix} (n-1)(n-2) & & (n+1)+(n-2)^2 \\ & \ddots & \\ (n+1)+(n-2)^2 & & (n-1)(n-2) \end{pmatrix}$$

$$\mathbf{A}^4(K_n) = \begin{pmatrix} (n-1)^2+(n-1)(n-2)^2 & & 2(n-1)(n-2)+(n-2)^3 \\ & \ddots & \\ 2(n-1)(n-2)+(n-2)^3 & & (n-1)^2+(n-1)(n-2)^2 \end{pmatrix}$$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)}-1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(n(n-1)^2 + n(n-1)(n-2)^2 - 2(n-1)(n-2)n - n(n-1)\right)$$

$$= \frac{n(n-1)}{8}\left((n-1) + (n-2)^2 - 2(n-2) - 1\right)$$

$$= \frac{n(n-1)}{8}\left(n - 1 + n^2 - 4n + 4 - 2n + 4 - 1\right)$$

$$= \frac{n(n-1)}{8}\left(n^2 - 5n + 6\right)$$

$$= \frac{n(n-1)(n-2)(n-3)}{8}$$

$$= \frac{n!}{8\,(n-4)!} \text{ simple 4-cycles}$$

There is a well known result that the number of simple m-cycles in K_n is given by $\frac{n!\,(k-1)!}{2\,k!\,(n-k)!}$ which for $k = 4$ matches our result.

[5 marks]

Answer 10

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 3 \end{pmatrix} \qquad \mathbf{A}^3 = \begin{pmatrix} 0 & 5 & 2 & 2 & 5 & 5 & 2 & 2 & 2 & 2 \\ 5 & 0 & 5 & 2 & 2 & 2 & 5 & 2 & 2 & 2 \\ 2 & 5 & 0 & 5 & 2 & 2 & 2 & 5 & 2 & 2 \\ 2 & 2 & 5 & 0 & 5 & 2 & 2 & 2 & 5 & 2 \\ 5 & 2 & 2 & 5 & 0 & 2 & 2 & 2 & 2 & 5 \\ 5 & 2 & 2 & 2 & 2 & 0 & 2 & 5 & 5 & 2 \\ 2 & 5 & 2 & 2 & 2 & 2 & 0 & 2 & 5 & 5 \\ 2 & 2 & 5 & 2 & 2 & 5 & 2 & 0 & 2 & 5 \\ 2 & 2 & 2 & 5 & 2 & 5 & 5 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 5 & 2 & 5 & 5 & 2 & 0 \end{pmatrix}$$

$$\mathbf{A}^4 = \begin{pmatrix} 15 & 4 & 9 & 9 & 4 & 4 & 9 & 9 & 9 & 9 \\ 4 & 15 & 4 & 9 & 9 & 9 & 4 & 9 & 9 & 9 \\ 9 & 4 & 15 & 4 & 9 & 9 & 9 & 4 & 9 & 9 \\ 9 & 9 & 4 & 15 & 4 & 9 & 9 & 9 & 4 & 9 \\ 4 & 9 & 9 & 4 & 15 & 9 & 9 & 9 & 9 & 4 \\ 4 & 9 & 9 & 9 & 9 & 15 & 9 & 4 & 4 & 9 \\ 9 & 4 & 9 & 9 & 9 & 9 & 15 & 9 & 4 & 4 \\ 9 & 9 & 4 & 9 & 9 & 4 & 9 & 15 & 9 & 4 \\ 9 & 9 & 9 & 4 & 9 & 4 & 4 & 9 & 15 & 9 \\ 9 & 9 & 9 & 9 & 4 & 9 & 4 & 4 & 9 & 15 \end{pmatrix}$$

$$\mathbf{A}^5 = \begin{pmatrix} 12 & 33 & 22 & 22 & 33 & 33 & 22 & 22 & 22 & 22 \\ 33 & 12 & 33 & 22 & 22 & 22 & 33 & 22 & 22 & 22 \\ 22 & 33 & 12 & 33 & 22 & 22 & 22 & 33 & 22 & 22 \\ 22 & 22 & 33 & 12 & 33 & 22 & 22 & 22 & 33 & 22 \\ 33 & 22 & 22 & 33 & 12 & 22 & 22 & 22 & 22 & 33 \\ 33 & 22 & 22 & 22 & 22 & 12 & 22 & 33 & 33 & 22 \\ 22 & 33 & 22 & 22 & 22 & 22 & 12 & 22 & 33 & 33 \\ 22 & 22 & 33 & 22 & 22 & 33 & 22 & 12 & 22 & 33 \\ 22 & 22 & 22 & 33 & 22 & 33 & 33 & 22 & 12 & 22 \\ 22 & 22 & 22 & 22 & 33 & 22 & 33 & 33 & 22 & 12 \end{pmatrix}$$

By Algorithm 2.2, the number of simple 5-cycles is given by,

$$\frac{1}{10}\left(tr\left(\mathbf{A}^5\right) - 5 \sum_{i=1}^{n} \left(\left(a_{ii}^{(3)} \right)\left(a_{ii}^{(2)} - 2\right)\right) - 5\, tr\left(\mathbf{A}^3\right)\right)$$

$$= \frac{1}{10}\left(120 - 0 - 0 \right)$$

$= 12$ simple 5-cycles

[5 marks]

Answer 11

(i) Each graph has 12 vertices. The “number of vertices” property has not detected any non-isomorphisms.

[1 mark]

(ii) Each graph has 18 edges. The “number of edges” property has not detected any non-isomorphisms.

[2 marks]

(iii) Each graph is 3-regular so the degree sequence of each is simply $\left(3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3\right)$. The “degree sequence” property has not detected any non-isomorphisms.

[2 marks]

(iv)

$$\mathbf{R} = \begin{pmatrix} 0&1&0&1&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&1&0&0&0&0&1&0 \\ 0&0&0&1&1&0&0&0&1&0&0&0 \\ 1&0&1&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&1&0&0&0&0&0&1 \\ 0&1&0&0&1&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&1&0&0&1&0&0&0&1&0 \\ 0&0&1&0&0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&1&1 \\ 0&1&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&1&0&0&0&1&0&0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0&1&0&0&0&0&1&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&1 \\ 0&1&0&1&0&0&0&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&0&0&1&0 \\ 0&0&0&1&0&1&0&0&0&1&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0&0&0&0 \\ 1&0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&1&0 \\ 0&0&1&0&1&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&1 \\ 0&1&0&0&0&1&0&0&0&0&1&0 \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} 0&1&0&0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&1&0&0 \\ 0&1&0&1&0&0&0&0&0&0&1&0 \\ 0&0&1&0&1&0&0&0&0&0&1&0 \\ 1&0&0&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&0&1 \\ 1&0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&0&1 \\ 0&1&0&0&0&0&0&0&1&0&1&0 \\ 0&0&1&1&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&1&0&1&0&0&0 \end{pmatrix}$$

[3 marks]

(v)

$$\mathbf{R}^3 = \begin{pmatrix} 0 & 5 & 1 & 6 & 2 & 0 & 6 & 1 & 1 & 2 & 2 & 1 \\ 5 & 0 & 2 & 1 & 1 & 5 & 1 & 2 & 2 & 1 & 5 & 2 \\ 1 & 2 & 0 & 5 & 5 & 1 & 2 & 1 & 5 & 1 & 2 & 2 \\ 6 & 1 & 5 & 0 & 0 & 2 & 1 & 6 & 2 & 2 & 1 & 1 \\ 2 & 1 & 5 & 0 & 2 & 5 & 1 & 1 & 1 & 2 & 2 & 5 \\ 0 & 5 & 1 & 2 & 5 & 2 & 1 & 1 & 2 & 2 & 1 & 5 \\ 6 & 1 & 2 & 1 & 1 & 1 & 0 & 6 & 5 & 1 & 2 & 1 \\ 1 & 2 & 1 & 6 & 1 & 1 & 6 & 0 & 2 & 1 & 5 & 1 \\ 1 & 2 & 5 & 2 & 1 & 2 & 5 & 2 & 0 & 5 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 5 & 0 & 5 & 5 \\ 2 & 5 & 2 & 1 & 2 & 1 & 2 & 5 & 1 & 5 & 0 & 1 \\ 1 & 2 & 2 & 1 & 5 & 5 & 1 & 1 & 1 & 5 & 1 & 2 \end{pmatrix}$$

$$tr\left(\mathbf{R}^3\right) = 6 \iff 1 \text{ triangle}$$

$$\mathbf{B}^3 = \begin{pmatrix} 2 & 5 & 0 & 1 & 1 & 2 & 5 & 5 & 1 & 2 & 2 & 1 \\ 5 & 0 & 5 & 1 & 3 & 1 & 2 & 1 & 3 & 0 & 1 & 5 \\ 0 & 5 & 0 & 6 & 0 & 3 & 1 & 2 & 1 & 6 & 2 & 1 \\ 1 & 1 & 6 & 0 & 6 & 1 & 1 & 1 & 2 & 1 & 5 & 2 \\ 1 & 3 & 0 & 6 & 0 & 5 & 0 & 2 & 1 & 6 & 1 & 2 \\ 2 & 1 & 3 & 1 & 5 & 0 & 5 & 1 & 3 & 0 & 1 & 5 \\ 5 & 2 & 1 & 1 & 0 & 5 & 2 & 5 & 1 & 2 & 2 & 1 \\ 5 & 1 & 2 & 1 & 2 & 1 & 5 & 2 & 5 & 0 & 0 & 3 \\ 1 & 3 & 1 & 2 & 1 & 3 & 1 & 5 & 0 & 5 & 5 & 0 \\ 2 & 0 & 6 & 1 & 6 & 0 & 2 & 0 & 5 & 0 & 2 & 3 \\ 2 & 1 & 2 & 5 & 2 & 1 & 2 & 0 & 5 & 2 & 0 & 5 \\ 1 & 5 & 1 & 2 & 1 & 5 & 1 & 3 & 0 & 3 & 5 & 0 \end{pmatrix}$$

$$tr\left(\mathbf{B}^3\right) = 6 \iff 1 \text{ triangle}$$

$$\mathbf{G}^3 = \begin{pmatrix} 0 & 5 & 1 & 1 & 5 & 2 & 5 & 1 & 3 & 0 & 3 & 1 \\ 5 & 0 & 6 & 3 & 1 & 1 & 0 & 2 & 0 & 6 & 1 & 2 \\ 1 & 6 & 2 & 5 & 2 & 1 & 1 & 0 & 2 & 1 & 6 & 0 \\ 1 & 3 & 5 & 2 & 5 & 0 & 1 & 1 & 1 & 2 & 5 & 1 \\ 5 & 1 & 2 & 5 & 0 & 5 & 2 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 5 & 0 & 1 & 7 & 0 & 2 & 1 & 7 \\ 5 & 0 & 1 & 1 & 2 & 1 & 0 & 7 & 0 & 3 & 0 & 7 \\ 1 & 2 & 0 & 1 & 1 & 7 & 7 & 0 & 7 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 2 & 0 & 0 & 7 & 0 & 5 & 0 & 7 \\ 0 & 6 & 1 & 2 & 2 & 2 & 3 & 0 & 5 & 0 & 6 & 0 \\ 3 & 1 & 6 & 5 & 1 & 1 & 0 & 1 & 0 & 6 & 2 & 1 \\ 1 & 2 & 0 & 1 & 1 & 7 & 7 & 0 & 7 & 0 & 1 & 0 \end{pmatrix}$$

$$tr\left(\mathbf{G}^3\right) = 6 \iff 1 \text{ triangle}$$

The “triangle count” property has not detected any non-isomorphisms.

[3 marks]

(vi)

The characteristic equations are:

For **R**:

$$\left(x+2\right)\left(x^4+x^3-4x^2-x+2\right)\left(x^5-7x^3+x^2+11x-4\right)$$

For **B**:

$$(x-3)(x+1)\left(x^2-2\right)\left(x^3+x^2-2x-1\right)\left(x^5+x^4-8x^3-3x^2+16x-6\right)$$

For **G**:

$$(x-3)\,x^2\left(x^9+3x^8-9x^7-29x^6+22x^5+82x^4-17x^3-77x^2+3x+13\right)$$

These show that none of the graphs are isomorphic to any of the others.

Lecture 3

Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

3.1 Subgraphs and Induced Subgraphs

The first two lectures focussed on the properties of a graph as an indivisible entity. Now the thinking shifts to pondering the patterns within a graph. We start with the idea of a *subgraph*.

Definition : Subgraph (Version 1)
A subgraph, G', of a graph G is a graph all of whose vertices are vertices of G and all of whose edges are edges of G.

For a given number of vertices, n, the number of possible unlabelled graphs, G_{UL}, increases rapidly with increasing n as the next table shows,

n	0	1	2	3	4	5	6	7	8	9	10	...
$n(G_{UL})$	1	1	2	4	11	34	156	1044	12346	274668	12005168	...

This is sequence A000088 in OEIS

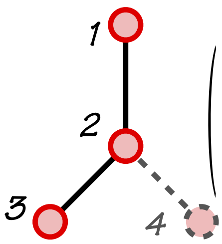
On top of this, even for a graph G with a small number of vertices, the number of non-isomorphic subgraphs is large.

Definition : Subgraph (Version 2)
A subgraph, G', can also defined as being a graph that is obtained from G by a sequence of edge and vertex deletions. When an edge is deleted it is simply removed but when a vertex is deleted also removed are all edges incident to it.

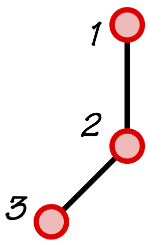
Definition : Induced Subgraph
A graph G' is said to be a (vertex) induced subgraph of G if G' can be obtained from G by a sequence of vertex-deletions only. That is, no edge-deletions are allowed other than those that occur as a part of a vertex-deletion.

3.2 Example of a (Vertex) Induced Subgraph

To the left is shown a graph along with its adjacency matrix. The vertex labelled *4* is about to be deleted along with any attached edge(s). This corresponds to deleting row 4 and column 4 in the adjacency matrix, The induced subgraph is on the right. The set of vertices {*1*, *2*, *3* } are termed the inducing vertices.



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \Rightarrow$$

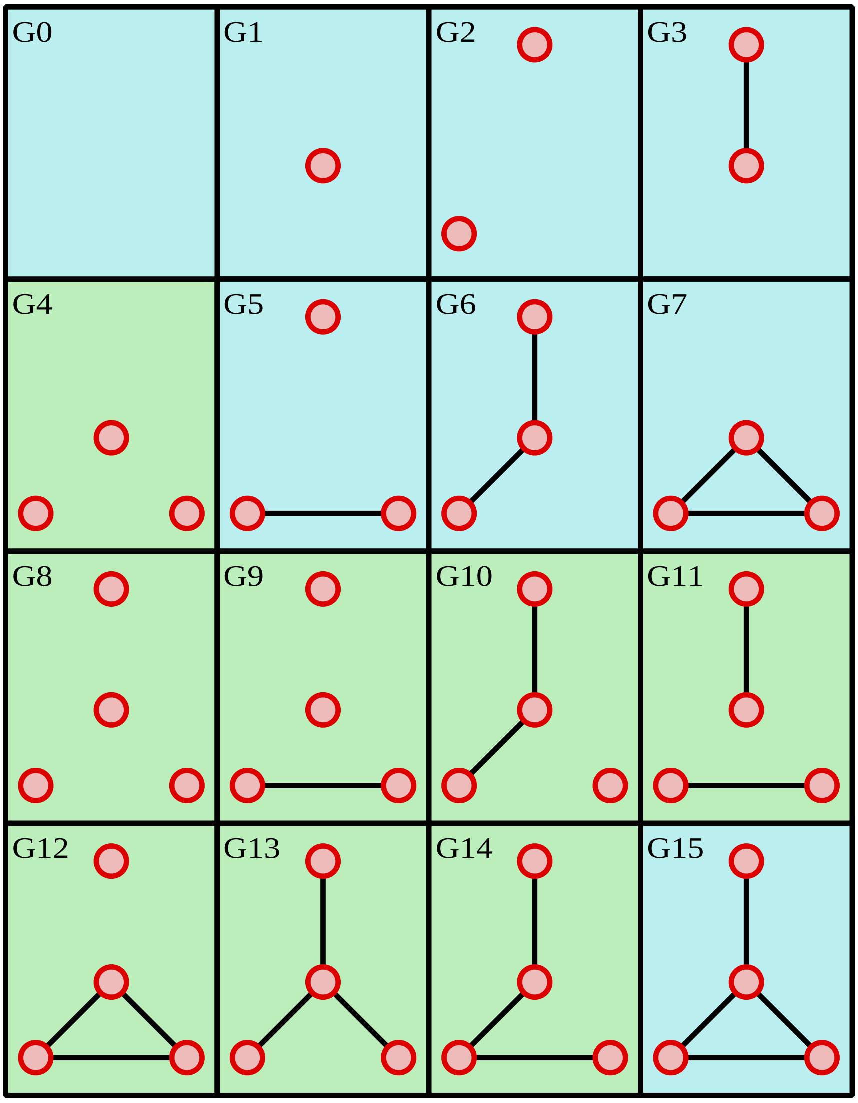


$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

3.3 Preserving Adjacency

As illustrated by the example, an induced subgraph keeps both adjacency and non-adjacency of the inducing vertices. In contrast, an ordinary subgraph preserves only non-adjacency.

The example looked at the graph G15 and one of its induced subgraphs. G15 is small enough for all of its subgraphs to be depicted and these are presented in the next diagram. Of the 16 possible subgraphs, 8 of them are (vertex) induced, and these are highlighted with a blue background.



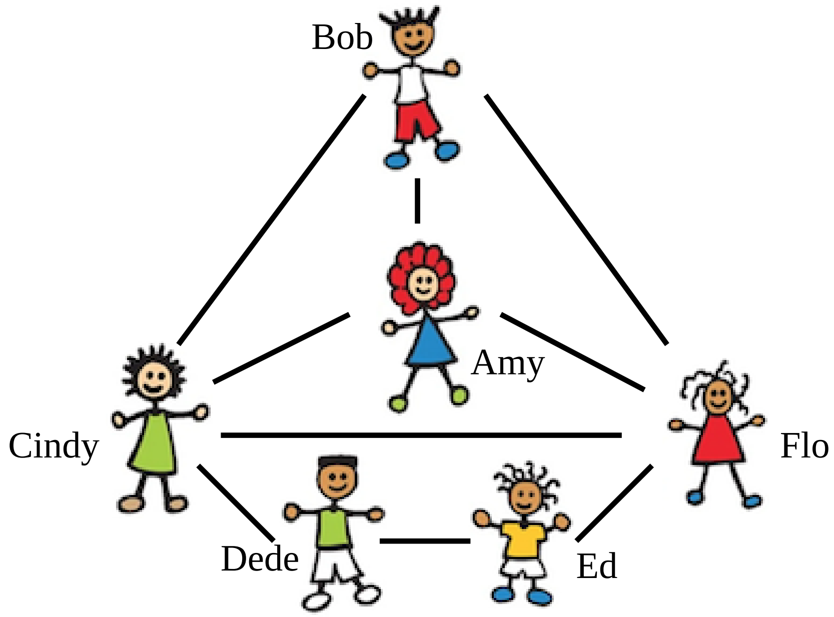
The 16 non-isomorphic subgraphs of G15, with the 8 that are induced in blue

3.4 The Null Graph

In the diagram the “null graph”, G0, is given as a subgraph although there is some debate amongst mathematicians on whether this empty graph is a valid entity or not. Many feel that it causes so much trouble and unnecessary issues that it is better to exclude it from Graph Theory altogether. Pragmatically, whether to include it as a valid and useful graph depends on context and convenience. In any theorem that involves the counting of graphs, check if the null graph is included.

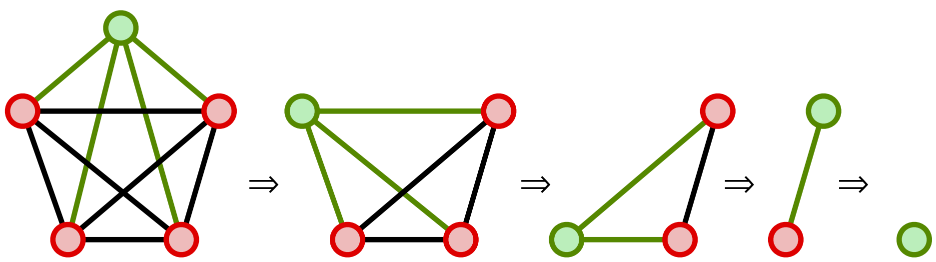
3.5 Clique and Maximal Clique

In a social group a collection of people who all know each other is referred to as a clique. This situation has an analogy in Graph Theory where the social group is the graph, and a clique is an induced subgraph that is complete.



The illustration shows who is friends with who in a class of six schoolchildren. Amy, Bob and Cindy form a clique because each is friends with the other two. Cindy, Dede, Ed and Flo do not form a clique because Ed and Cindy are not friends, for example. Although Amy, Bob and Cindy form a clique, their clique is not maximal because Flo can be added into the friendship group and each possible pairing amongst the four is a friendship relation. The clique of Amy, Bob, Cindy and Flo is now maximal because no further child can be added to the four that would result in a larger clique. Notice that Dede and Ed, for example, also form a maximal clique as each member of that clique knows all the others, and no other child can be added that would result in a larger clique.

The following diagram illustrates the fact that any vertex induced subgraph of a complete graph forms a clique.



K_5 K_4 K_3 K_2 K_1

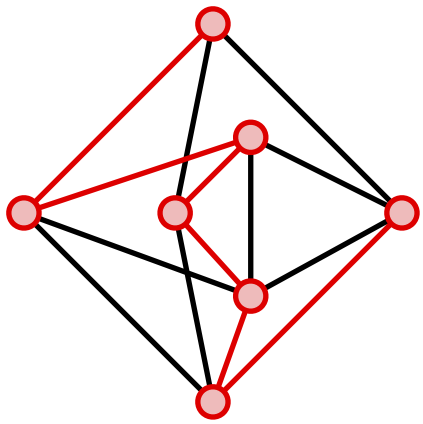
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow (0)$$

3.6 Some Other Pieces of Graph

Having dealt briefly with the subgraphs of a graph, we return to this lecture's main theme which was “patterns within a graph”. A crucial next step is to identify three entities known as path, trail and walk.

Definition : Path (of Distinct Vertices)

A path on n distinct vertices in a graph G is denoted, P_n. Discussed in section 1.3, a path will have two end vertices of degree at least 1 and thread its way through each of the remaining $(n - 2)$ vertices which will each be of degree at least 2. To emphasise; a path with at least one edge cannot be closed, that is, the first and last vertex cannot be the same vertex.



A path, P_7, (in red) on 7 vertices in graph G1169

Definition : Trail (of Distinct Edges)

A trail in a graph G is a (possibly empty) sequence of distinct edges in G such that any two consecutive edges in the sequence are incident to a common vertex. A trail allows repetitions of the same vertex whereas a path does not. If a non empty trail has only the first and last vertices equal it is termed a *cycle* or closed trail. If, in addition to having the same first and last vertex, it has other vertices occurring more than once it is termed a *circuit.*

Definition : Walk

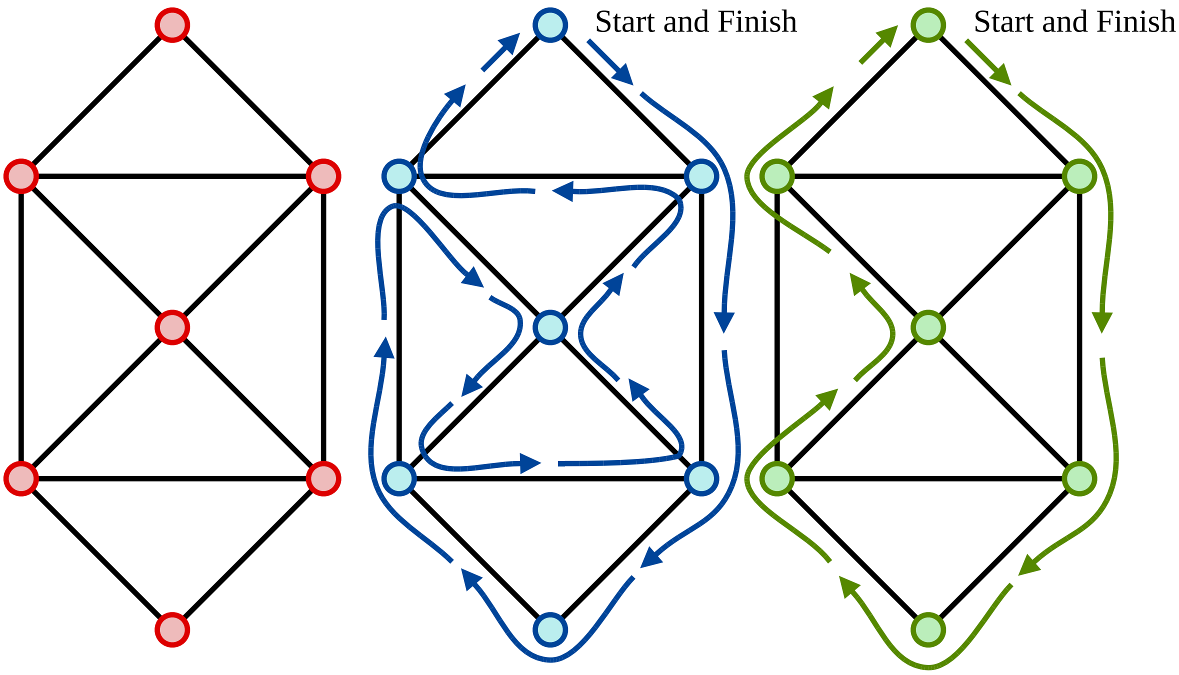
A walk on a graph G drops the requirement that the edges in the sequence of edges be distinct. A *closed walk* is a walk where the start and the finish are the same vertex.

3.7 The Eularian Edge Traveller

Given a connected graph, we can unleash upon it the Eularian Edge Traveller. This can be thought of as a spider that attempts to find a way to crawl along a closed trail that traverses each and every edge in the graph once and once only, starting and finishing at the same vertex. If such a closed trail can be found the graph is said to be *Eularian* and the trail said to be a *Eulerian trail***.**

For example, the graph below left (with red vertices) is Eularian and by way of demonstrating this a Eularian trail is shown in the central diagram (with blue vertices) threading its way through each edge once and once only.

Note that in this particular example there are other valid closed trails (even from the same start/finish vertex) that could have been used show that the graph is Eularian.



3.8 The Hamiltonian Vertex Visitor

Given a connected graph, we can unleash upon it the Hamiltonian Vertex Visitor. This can again be thought of as a spider. This time, however, it attempts to find a way to crawl in a cycle that passes through every vertex in the graph once and once only. If such a cycle can be found the graph is said to be Hamiltonian and the cycle is said to be a Hamiltonian cycle.

For example, the graph above left (with red vertices) is Hamiltonian. The third diagram in the illustration (with green vertices) shows this by exhibiting a closed trail that visits each vertex exactly once.

Note that in this particular example there are other valid cycles that could have been used to show that the graph is Hamiltonian.

3.9 Proofs

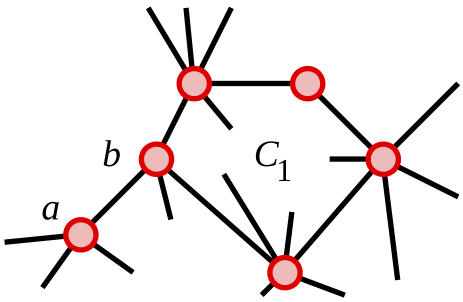
In this final section, we work our way through a sequence of "classic" elementary graph theory proofs to do with the connectivity of graphs.

Lemma 3.1 : On the Degree of the Vertices of an Eulerian Graph
If G is an Eulerian graph, then each vertex of G has even degree.

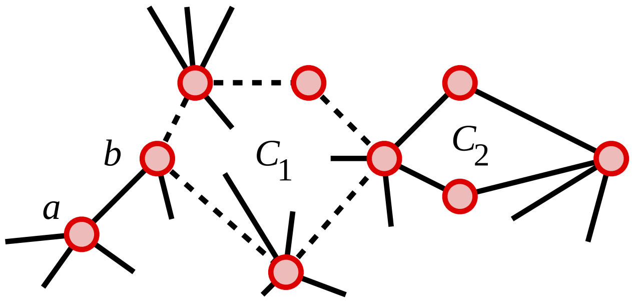
Proof
If G is Eulerian, then there is an Eulerian trail, that is, a closed trail that traverses each and every edge in the graph once and once only. Whenever this trail passes through a vertex there is a contribution of +2 to the degree of that vertex. Since each edge is used just once, the degree of each vertex is a sum of 2s. But that is precisely what an even number is. Thus, each vertex is of even degree. ☐

Lemma 3.2 : Eulerian Graph Decomposition
A graph that is Eulerian (and therefore connected) can be decomposed into distinct cycles, no two of which have an edge in common.

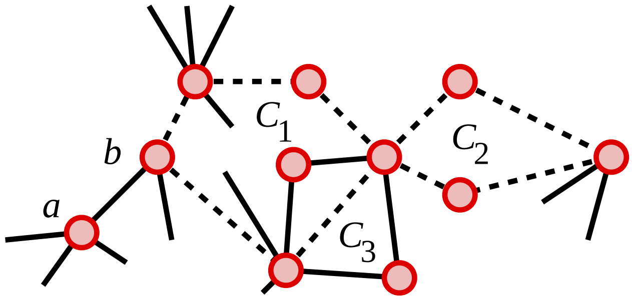
Proof
The first cycle is obtained by starting at any vertex a and traversing edges in an arbitrary manner, never repeating any edge. As the graph is Eulerian, each vertex is of even degree, by Lemma 3.1, which guarantees that whenever a vertex is entered, there is a way out. With only a finite number of vertices in G, eventually a vertex b is reached that has been previously encountered.



The edges of the closed trail between the two occurrences of the vertex b form a cycle, C_1 which is now removed from G, leaving a (possibly disconnected) graph H in which each vertex has even degree. If G is simply C_1 then we are done. If not then the procedure is repeated to find in H a cycle C_2 (which will not have any edges in common with C_1).



Removing the edges of C_2 from H leaves yet another graph in which each vertex has even degree, and which therefore contains a cycle C_3.



By continuing in this manner there will eventually be no edges left but, along the way, a set of distinct cycles C_1, C_2, C_3, ..., C_k have been obtained that include every edge of G, and no two of which have an edge in common. □

In Lemma 3.2 we began with the fact that a graph was Eularian to reason that each vertex was of even degree (by Lemma 3.1) and then deduced that the graph could be decomposed into distinct cycles. However, if we drop the initial requirement that the graph be Eularian, the argument employed in proving Lemma 3.2 will still apply from the starting point of having a graph with vertices all of even degree. Such a graph can still be decomposed into distinct cycles. Must it then be Eularian ? Our next lemma, Lemma 3.3, states that it is.

Lemma 3.3 : Graphs With Vertices All Even

If each vertex of a connected graph G has even degree, then G is Eulerian.

Proof

As just discussed, a connected graph in which each vertex is of even degree can be decomposed into distinct cycles. To now show this graph is Eularian the cycles or, rather, parts of the cycles, must be fitted together in some way to make an Eularian trail. Here is an algorithm that will do this; start at any vertex of the cycle C_1 and travel round C_1 until the vertex of another cycle is encountered such as, for example, C_2. Now pass along the edges of this cycle before resuming the journey around C_1, traversing other new cycles as they are encountered. At this juncture a closed trail that includes C_1 and (unless G is just C_1) at least one other cycle (since G is connected) has been traversed. If this trail includes all of the cycles C_1, C_2, C_3, ..., C_k then that is the required Eularian trail. If not, a journey is commenced round the new closed trail, traversing other cycles as they are met. There will always be at least one cycle to add to the previous closed trail because G is connected. The process is repeated in this manner until all the cycles have been traversed, whereupon the sought after Eularian trail is obtained. From this is deduced that G is Eularian. □

Theorem 3.1 : Eularian IFF

A connected graph is Eularian if and only if each vertex has even degree.

Proof

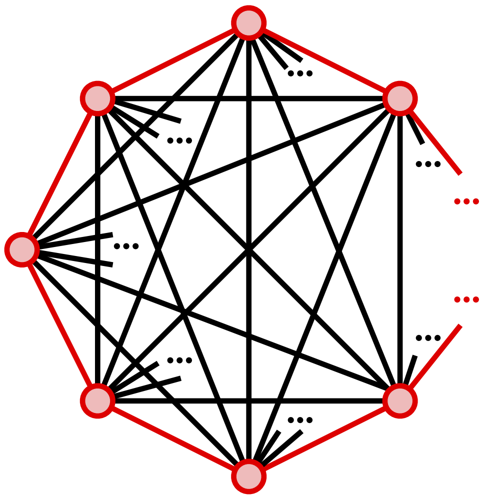
Lemma 3.1 and Lemma 3.2 in combination prove this theorem. □

Lemma 3.4 : K_n is Hamiltonian

The complete graph, K_n, is Hamiltonian for all values of n.

Proof

The complete graph K_n for any given value of n can be drawn with one vertex at each vertex of an n-gon as illustrated by the following diagram,



The perimeter of the diagram, highlighted in red, is the cycle subgraph C_n of K_n which is a cycle that includes every vertex.

This shows that K_n contains a Hamiltonian cycle for all values of n and so, by definition, is Hamiltonian. □

Theorem 3.2 A Test For Hamiltonianism (Ore, 1960)

If G is a simple graph with n ($\geqslant 3$) vertices, and if $deg(v) + deg(w) \geqslant n$ for each pair of non-adjacent vertices v and w, then G is Hamiltonian.

Proof (by contradiction)

Suppose, for a contradiction, that G is a simple graph with n ($\geqslant 3$) vertices, and with $deg(v) + deg(w) \geqslant n$ for each pair of non-adjacent vertices v and w but that G is not Hamiltonian. Pick any two vertices of G which are not already joined by an edge and add a new edge between them. Keep on doing this until a graph G_{Last} is obtained which, for the first time, does have a closed Hamiltonian path. (This event must eventually occur because the complete graph on n vertices acts as a "catching feature" because, by Lemma 3.4, the graph K_n is Hamiltonian).

Let G_{ButOne} be the graph immediately prior to G_{Last}, let $\{x, y\}$ be the edge added to G_{ButOne} to obtain G_{Last} and let $(z_1, z_2, ..., z_n, z_1)$ be a Hamiltonian cycle in G_{Last}. This must use the edge $\{x, y\}$ somewhere in the cycle because otherwise G_{ButOne} would have had a cycle (which would contradict the earlier requirement that it did not).

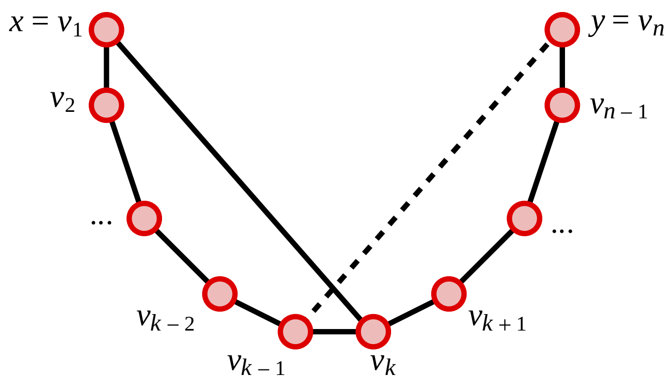
There are now two cases to consider;

Case 1 :

If $\{z_n, z_1\} = \{x, y\}$ then $(z_1, z_2, ..., z_n)$ is a non-closed Hamiltonian path in G_{ButOne}. Relabel the path's vertices $(v_1, v_2, ..., v_n)$ such that $x = v_1$ and $y = v_n$.

Case 2 :
If $\{z_n, z_1\} \neq \{x, y\}$ then there must be some r such that $1 \leqslant r < n$ and $\{x, y\} = \{z_r, z_{r+1}\}$ and $(z_{r+1}, ..., z_n, z_1, ..., z_r)$ is a non-closed Hamiltonian path in G_{ButOne}. Relabel the path's vertices $(v_1, v_2, ..., v_n)$ such that $x = v_1$ and $y = v_n$.

In both cases we emerge with a non-closed Hamiltonian path that is labelled $(v_1, v_2, ..., v_n)$ with $x = v_1$ and $y = v_n$ as illustrated below.



The focus now is upon the condition $deg(v) + deg(w) \geqslant n$.
The key observation is to note that for any vertex in the path that x is adjacent to, (say v_k where $2 \leqslant k \leqslant n$), y cannot be adjacent to the preceding vertex in the path (say v_{k-1} where $2 \leqslant k \leqslant n$) because the edge $\{x, y\}$ was specifically removed when constructing G_{ButOne} from G_{Last}.
Suppose for the sake of argument that y was adjacent to the preceding vertex as shown with a broken line. This cannot be the case because if it were then we would have the following Hamiltonian cycle,

$$x, v_2, ..., v_{k-2}, v_{k-1}, y, v_{n-1}, ..., v_{k+1}, v_k, x$$

Notice that this is only an issue if y is adjacent to the preceding vertex; for any other vertex a Hamiltonian cycle does not result.

Consequently, $deg(y) \leqslant n - 1 - deg(x)$ giving $deg(x) + deg(y) \leqslant n - 1$ which contradicts the earlier condition that $deg(v) + deg(w) \geqslant n$ for each pair of non-adjacent vertices v and w. □

The following teaching video from "Wrath of Math" goes through a proof of Ore's theorem along similar lines to that presented above;

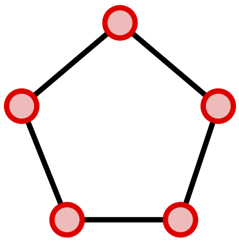
Teaching Video: http://www.NumberWonder.co.uk/v9119/3.mp4

3.10 Exercise

Marks Available : 60

Question 1

In section 1.3 a *cycle graph* was defined as being a graph that consists of a single cycle of vertices and edges and denoted C_n. The example of C_5 was given as;

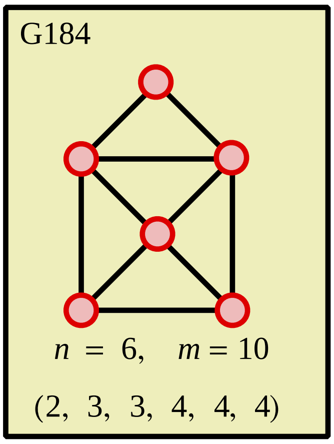


Charles claims that, "If every vertex of a simple graph G has degree 2, then G is a cycle". Exhibit a counterexample on six vertices that proves Charles is wrong.

[2 marks]

Question 2

(i) By first quoting a lemma or theorem, explain why G184 is not Eulerian.



[1 mark]

(ii) Although not Eulerian, the graph is semi-Eulerian.
This means that there exists a (non-closed) trail that includes every edge. Annotate the graph with such a trail.

[2 marks]

Question 3
Give an example of a connected graph on five vertices that is non-Eulerian (neither Eularian nor semi-Eularian) that contains a cycle.

[2 marks]

Question 4
Prove the following lemma;

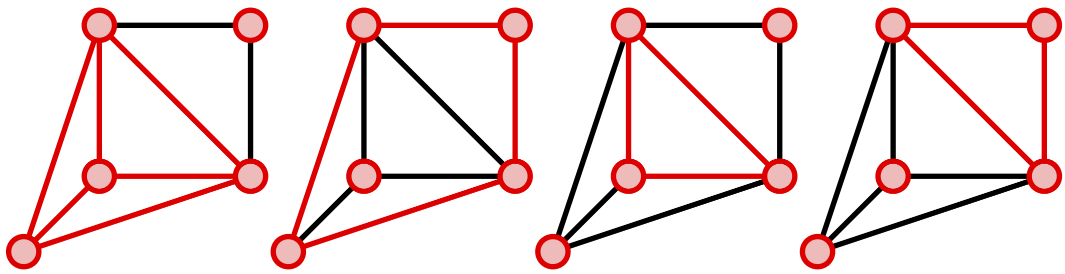
Lemma 3.5 : Must Contain A Cycle
Any finite, connected graph, with more than two vertices, in which every vertex is of degree of at least 2, must contain a cycle.

[4 marks]

Question 5

In each of the following four graphs, a part of the graph is highlighted in red.
Describe each of the highlighted configurations with one of the following;

(a) "maximal clique"
(b) "non-maximal clique"
(c) "not a clique"



[4 marks]

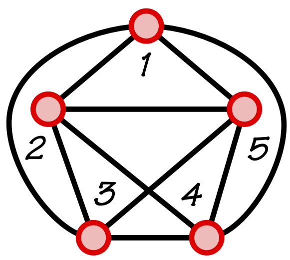
Question 6

The graph depicted below is of the complete graph, K_5.

It contains a trail that traverses each edge once and once only, with all vertices being encountered more than once.

Such a trail is, for example, 1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1

This is an example of an Euler circuit.



Explain why K_n does not contain an Euler circuit when n is even.

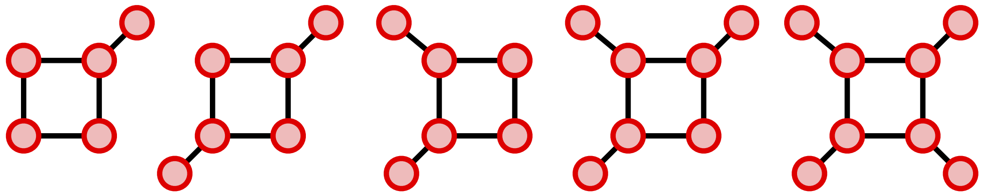
[3 marks]

Question 7

Give an example on four vertices of a connected graph that has no Hamilton path.

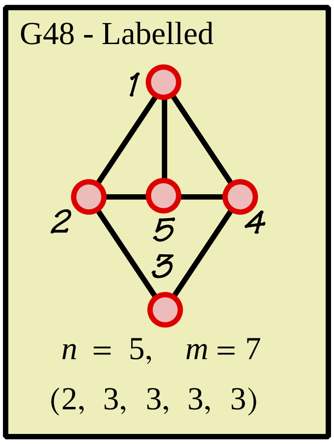
[2 marks]

Question 8
For each of these connected graphs state if they have a Hamiltonian path or not.



[3 marks]

Question 9
Show how Ore's Theorem correctly predicts that the following graph is Hamiltonian, and then annotate the graph to show such a cycle.



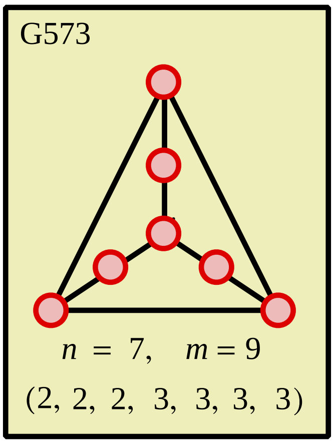
[3 marks]

Question 10
Show that the cycle graph, C_5, is Hamiltonian in spite of Ore's Theorem not being satisfied.

[3 marks]

Question 11

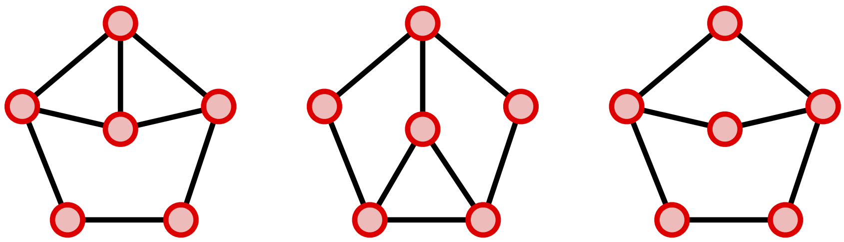
Explain why the following graph is not Hamiltonian.



[3 marks]

Question 12

A “Uniquely Hamiltonian Graph” is a graph possessing a single Hamiltonian cycle.
Determine which, if any, of the following are Uniquely Hamiltonian Graphs.
In each case, give a reason for your answer.



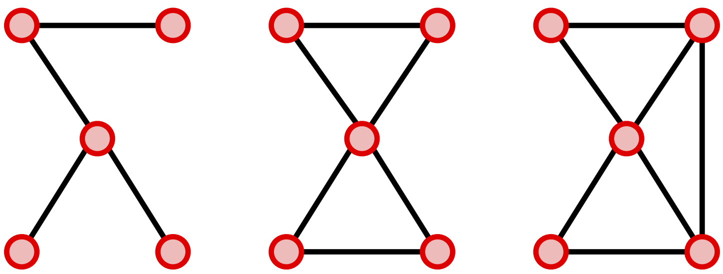
[3 marks]

Question 13

Determine if each of the following graphs is

(i) Eulerian

(ii) Hamiltonian



[3 marks]

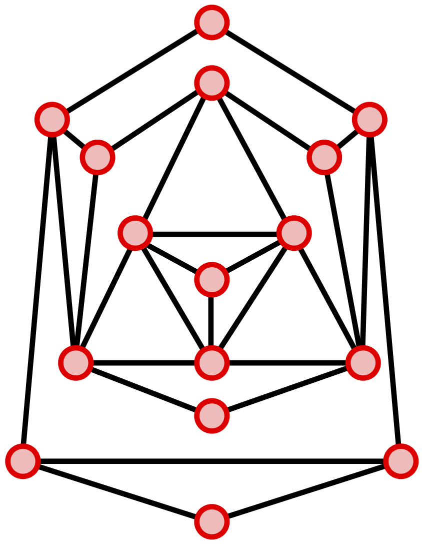
Question 14

Determine if the following graph is,

(i) Eulerian

(ii) Hamiltonian

Give a reason for each of your answers.

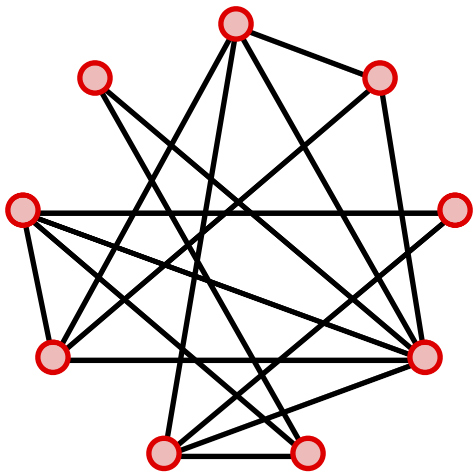


[4 marks]

Question 15

The graph represents friendships between a group of students where each vertex is a student and each edge is a friendship. Is it possible for the students to sit around a round table in such a way that every student sits between two friends ?

(With thanks to Oscar Levin for this question)



[4 marks]

Question 16

Prove the following corollary to Ore's Theorem,

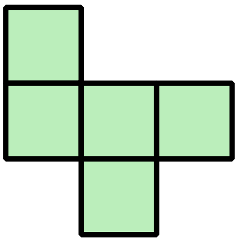
Corollary 3.1 : Dirac's Theorem

If G is a simple connected graph with n vertices, where $n \geqslant 3$, and $deg(v) \geqslant \dfrac{n}{2}$ for each vertex v, then G is Hamiltonian.

Hint : Use Ore's Theorem !

[4 marks]

Question 17

Consider the following pentomino, 

(i) Draw a graph G of this pentomino with five vertices and 4 edges.

[1 mark]

(ii) Draw a distinct pentomino whose graph is isomorphic to G.

[1 mark]

(iii) Draw two distinct pentominos whose graphs are not isomorphic to G, not isomorphic to each other, and not isomorphic to the path, P_5

[3 marks]

Question 18

In answering question 3, you will have found that the vertices of odd degree played a key role; one being the start of the (non-closed) trail and the other the finish. With that in mind the following lemma will not come as a surprise.

Lemma 3.6 : Semi-Eularian IFF

A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

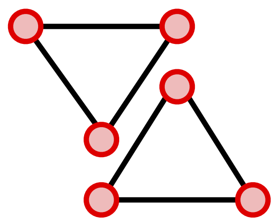
Prove lemma 3.4.

[5 marks]

3.11 Answers to 3.10 Exercise

Answer 1

Charles forgot to state that the graph must be connected for his statement to be true. The counterexample to his statement without the word connected is;



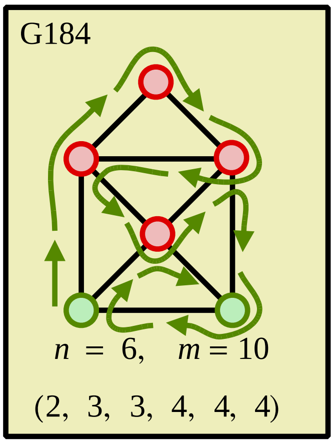
[2 marks]

Answer 2

(i) The degree sequence of G184, given in the diagram, is (2, 3, 3, 4, 4, 4). By Theorem 3.4, "a connected graph is Eularian if and only if each vertex has even degree" and so the odd numbers in the degree sequence mean G184 is not Eularian.

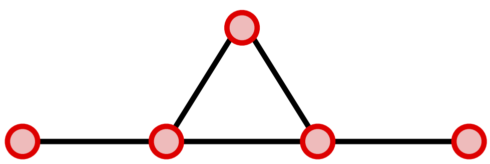
[1 mark]

(ii) The trail will have to start at one of the vertices of odd degree and finish at the other vertex of odd degree, and pass along each and every edge once. One such is depicted below but there are others.



[2 marks]

Answer 3



[2 marks]

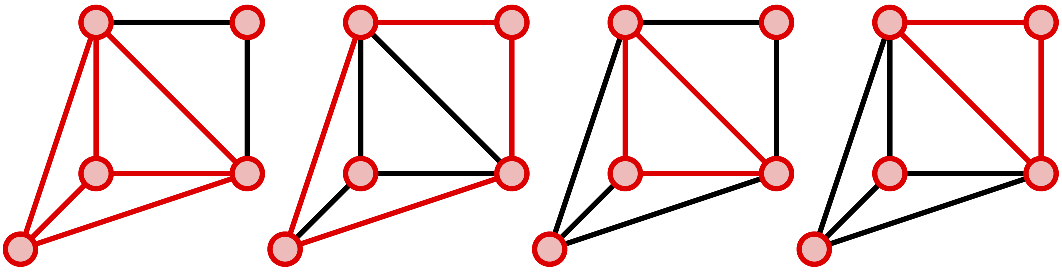
Answer 4

Lemma 3.5 : Must Contain A Cycle
Any finite, connected graph, with more than two vertices, in which every vertex is of degree of at least 2, must contain a cycle.

Proof (from "Introduction to Graph Theory by Robin J Wilson")
We begin by assuming that the graph, G, is simple.
(If it were not, with loops or multiple edges, the result is trivial)
Let v be any vertex of G and construct a walk v, v_1, v_2, ... inductively, by choosing v_1 to be any vertex adjacent to v and, for each $k > 1$, choosing v_{k+1} to be any vertex adjacent to v_k except v_{k-1}, the existence of such a vertex guaranteed by the hypothesis. Since G has only finitely many vertices, eventually a vertex will be chosen that has been chosen before. Let v_c be the first such vertex in which case the part of the walk that lies between the two occurrences of v_c is the required cycle. □

[4 marks]

Answer 5



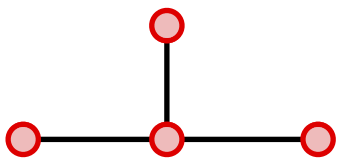
maximal clique　　not a clique　　non-maximal clique　　maximal clique

[4 marks]

Answer 6
Theorem 3.1 points out that "A connected graph is Eularian if and only if each vertex has even degree". To have a Eularian Circuit the graph has to first be Eularian. When n is even, K_n is $(n - 1)$-regular, and so all vertices are of odd degree. Thus there can be no Eularian Circuit when n is even.

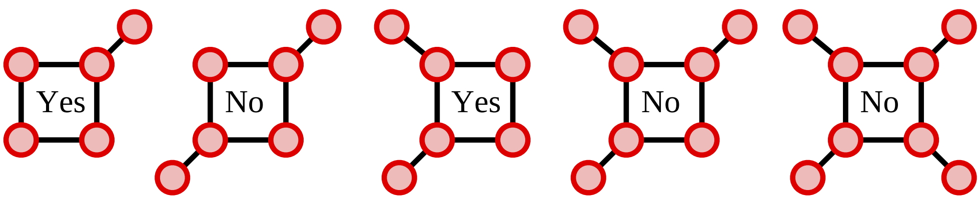
[3 marks]

Answer 7



[2 marks]

Answer 8



[3 marks]

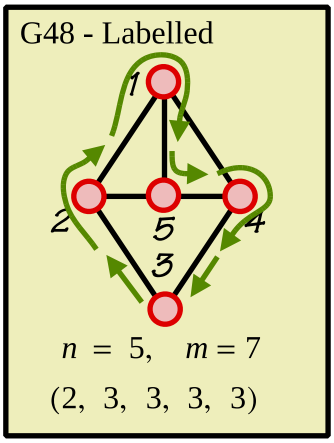
Answer 9

The non-adjacent vertices with their degree sum are,

13 : $3 + 2 = 5$

24 : $3 + 3 = 6$

35 : $2 + 3 = 5$

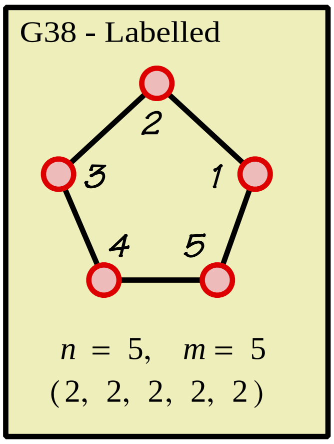


Ore's Theorem requires that all the degree sums are at least 5 which is so.
Thus G48 is Hamiltonian.
The annotation shows one of the two possible Hamiltonian cycles.

[3 marks]

Answer 10

With the C_5 graph labelled as shown, the non-adjacent vertices with their degree sum are,



13 : $2 + 2 = 4$

14 : $2 + 2 = 4$

24 : $2 + 2 = 4$

25 : $2 + 2 = 4$

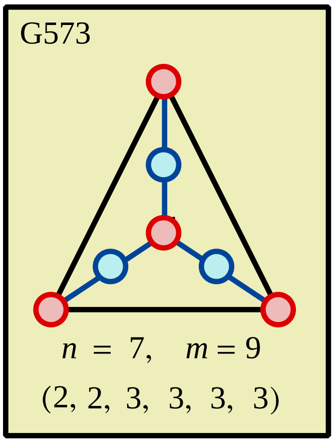
35 : $2 + 2 = 4$

Ore's theorem is not satisfied because it requires that all of the degree sums must be at least 5 (the number of vertices) which is not so (none of them are).
In spite of this the graph is obviously Hamiltonian.
We say that Ore's Theorem is a sufficient condition but not a necessary one.

[3 marks]

Answer 11

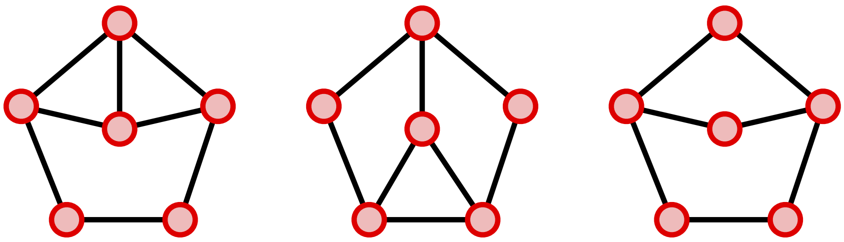
Any vertex of degree two has to be on the Hamiltonian cycle and the edges incident to such a vertex must be traversed. The vertices of degree two and their incident edges are coloured blue in the following annotation of G573,



All three edges that are incident to the centre vertex thus would have to be on any Hamiltonian cycle which means that such a cycle cannot exist.

[3 marks]

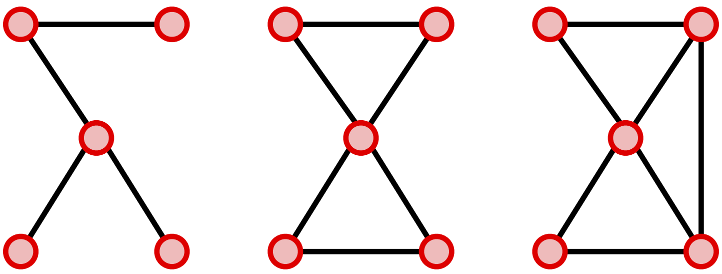
Answer 12



The leftmost graph is Hamiltonian but not unique as there are two distinct Hamiltonian cycles that can be drawn, each missing out a different edge.
The central graph is uniquely Hamiltonian, the cycle missing out the vertical edge.
The rightmost graph is not Hamiltonian at all and so not uniquely Hamiltonian.

[3 marks]

Answer 13



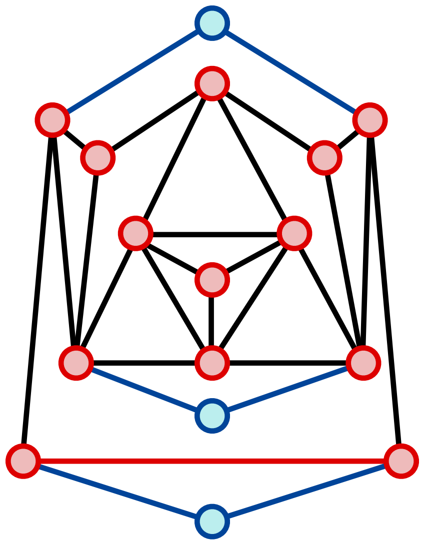
The leftmost graph is neither Eularian nor Hamiltonian.
The central graph is Eularian but not Hamiltonian.
The rightmost graph is not Eularian but it is (uniquely) Hamiltonian.

[3 marks]

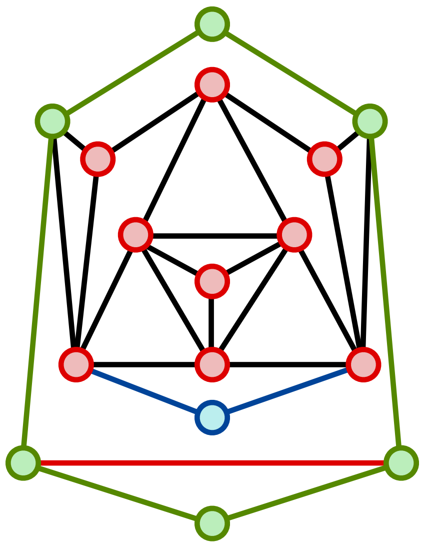
Answer 14

(i) By Theorem 3.1, "A connected graph is Eularian if and only if each vertex has even degree". Many of the vertices in the graph are of odd degree and so it is not Eulerian.

(ii) Any vertex of degree two has to be on the Hamiltonian cycle and the edges incident to such a vertex must be traversed. The vertices of degree two and their incident edges are coloured blue in the following annotation the given graph,



Of these it is the lowest vertex that is problematical, and results in the edge highlighted in red being excluded from any Hamiltonian cycle. This in turn means that the two side edges must be in the cycle. However, we that have the six cycle highlighted in green below as having to be a part of the Hamiltonian path.

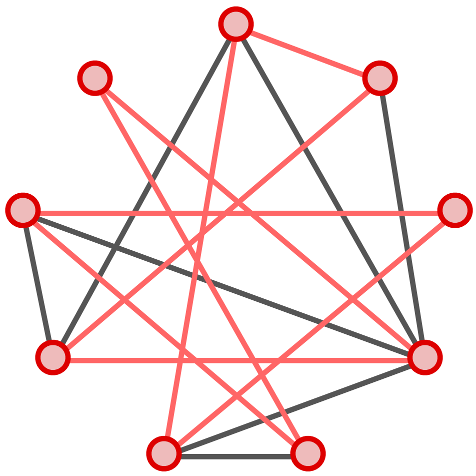


However, on looking at the way this six cycle joins to the rest of the graph it becomes clear that it cannot be a part of a Hamiltonian cycle. Thus the graph is not Hamiltonian.

[4 marks]

Answer 15

The graph needs searching to see if it has a Hamiltonian cycle.
It does !



[4 marks]

Answer 16

Corollary 3.1 : Dirac's Theorem

If G is a simple connected graph with n vertices, where $n \geqslant 3$, and $deg\,(v) \geqslant \frac{n}{2}$ for each vertex v, then G is Hamiltonian.

Proof

Suppose that we have a graph that satisfies the conditions of the theorem and which is not complete, for if it were, it would be Hamiltonian and we'd be done. Take two non-adjacent vertices, u and v, from the graph G that are not joined by an edge. Consider the sum of the degrees of these vertices, $deg\,(u) + deg\,(v)$.

From the theorem's conditions we know that $deg\,(u) \geqslant \frac{n}{2}$ and $deg\,(v) \geqslant \frac{n}{2}$.

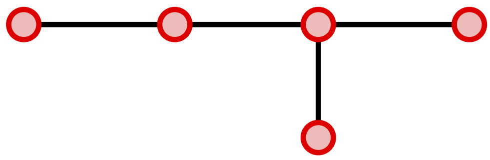
Then $deg\,(u) + deg\,(v) \geqslant \frac{n}{2} + \frac{n}{2}$ and we have that, $deg\,(u) + deg\,(v) \geqslant n$.

From Ore's Theorem, G is Hamiltonian. ☐

[4 marks]

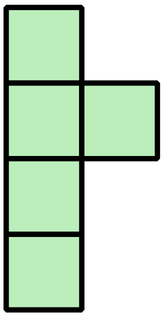
Answer 17

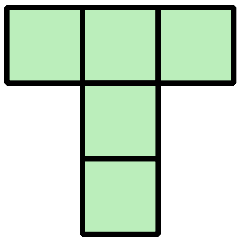
(i) Anything isomorphic to,



[1 mark]

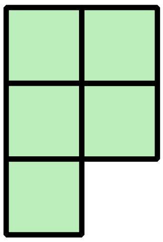
(ii) Either one of the following two pentominos,

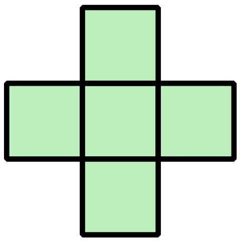




[1 mark]

(iii)





[3 marks]

Answer 18

Lemma 3.6 : Semi-Eularian IFF
A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Proof

(⇒)

If G is semi-Eulerian then there is an open Euler trail, T, in G. Suppose the trail begins at v_1 and end at v_n. Except for the initial occurrence of v_1 and the concluding occurrence of v_n, each time a vertex is encountered, it accounts for two edges adjacent to that vertex, the one before it in the trail and the one after. T uses every edge exactly once. So every edge is accounted for without repetition. In conclusion, the degree of every vertex must be even except for v_1 and v_n which must both be odd.

(⇐)

Suppose u and v are the two vertices of odd degree. Consider the related graph G' where a single edge has been added to G between u and v. Every vertex in G' is of even degree and so by Theorem 3.1, "A connected graph is Eularian if and only if each vertex has even degree", G' has a closed Euler trail. This closed trail must use the edge between u and v. Thus there must be an open Euler trail in G when the edge between u and v is removed.

□