Lecture 3

Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

3.1 Subgraphs and Induced Subgraphs

The first two lectures focussed on the properties of a graph as an indivisible entity. Now the thinking shifts to pondering the patterns within a graph. We start with the idea of a *subgraph*.

Definition : Subgraph (Version 1)
A subgraph, G', of a graph G is a graph all of whose vertices are vertices of G and all of whose edges are edges of G.

For a given number of vertices, n, the number of possible unlabelled graphs, G_{UL}, increases rapidly with increasing n as the next table shows,

n	0	1	2	3	4	5	6	7	8	9	10	...
$n(G_{UL})$	1	1	2	4	11	34	156	1044	12346	274668	12005168	...

This is sequence A000088 in OEIS

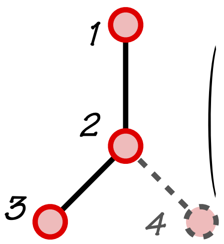
On top of this, even for a graph G with a small number of vertices, the number of non-isomorphic subgraphs is large.

Definition : Subgraph (Version 2)
A subgraph, G', can also defined as being a graph that is obtained from G by a sequence of edge and vertex deletions. When an edge is deleted it is simply removed but when a vertex is deleted also removed are all edges incident to it.

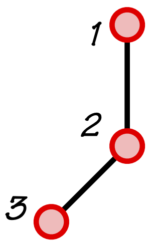
Definition : Induced Subgraph
A graph G' is said to be a (vertex) induced subgraph of G if G' can be obtained from G by a sequence of vertex-deletions only. That is, no edge-deletions are allowed other than those that occur as a part of a vertex-deletion.

3.2 Example of a (Vertex) Induced Subgraph

To the left is shown a graph along with its adjacency matrix. The vertex labelled *4* is about to be deleted along with any attached edge(s). This corresponds to deleting row 4 and column 4 in the adjacency matrix, The induced subgraph is on the right. The set of vertices {*1*, *2*, *3* } are termed the inducing vertices.



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \Rightarrow$$

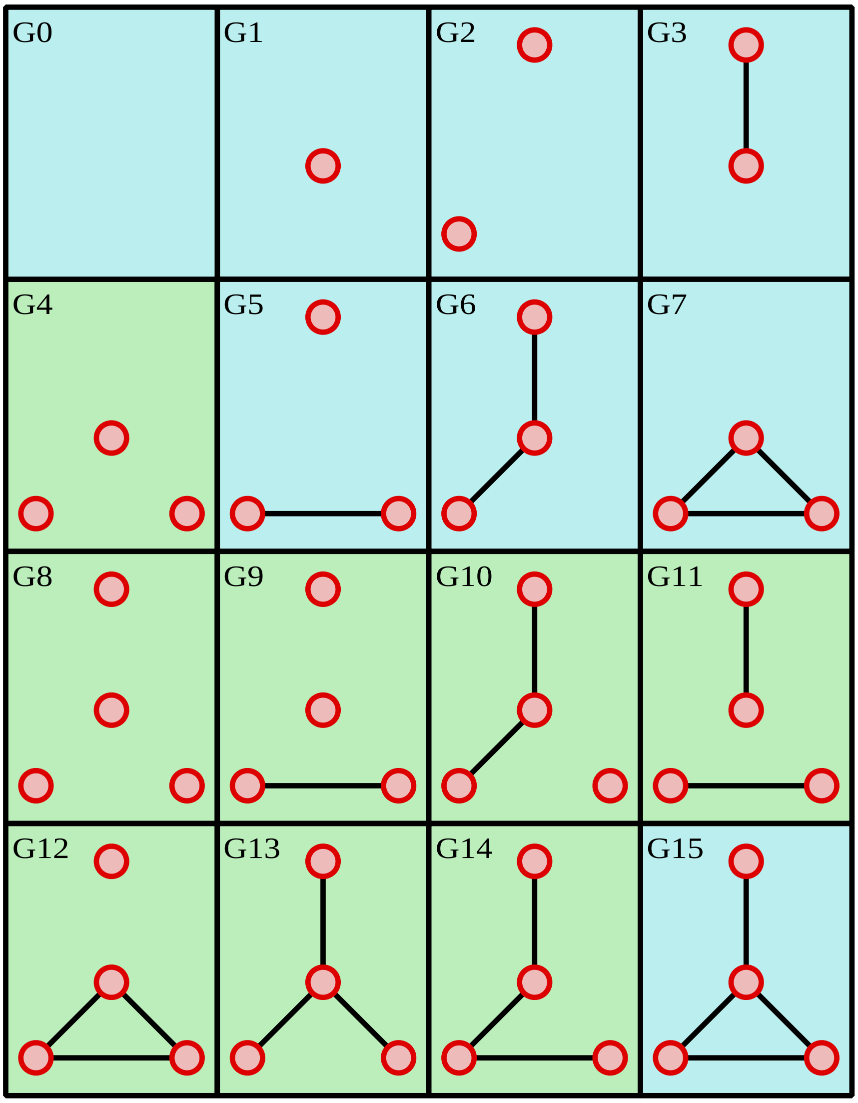


$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

3.3 Preserving Adjacency

As illustrated by the example, an induced subgraph keeps both adjacency and non-adjacency of the inducing vertices. In contrast, an ordinary subgraph preserves only non-adjacency.

The example looked at the graph G15 and one of its induced subgraphs. G15 is small enough for all of its subgraphs to be depicted and these are presented in the next diagram. Of the 16 possible subgraphs, 8 of them are (vertex) induced, and these are highlighted with a blue background.



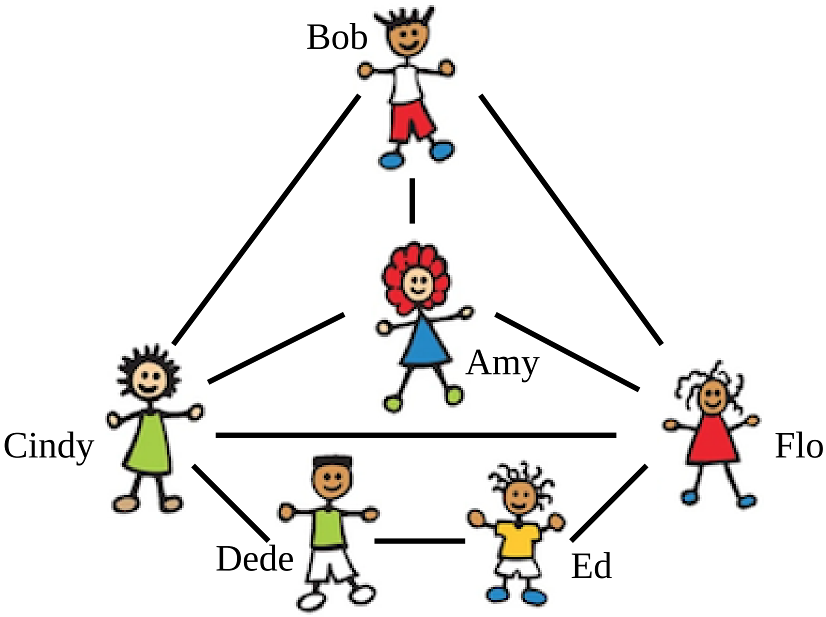
The 16 non-isomorphic subgraphs of G15, with the 8 that are induced in blue

3.4 The Null Graph

In the diagram the “null graph”, G0, is given as a subgraph although there is some debate amongst mathematicians on whether this empty graph is a valid entity or not. Many feel that it causes so much trouble and unnecessary issues that it is better to exclude it from Graph Theory altogether. Pragmatically, whether to include it as a valid and useful graph depends on context and convenience. In any theorem that involves the counting of graphs, check if the null graph is included.

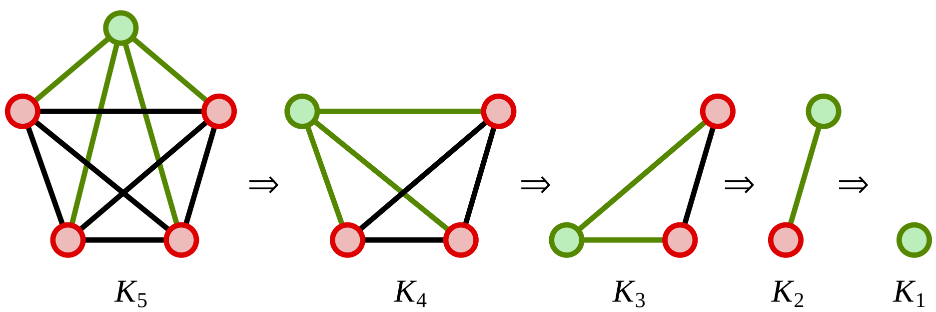
3.5 Clique and Maximal Clique

In a social group a collection of people who all know each other is referred to as a clique. This situation has an analogy in Graph Theory where the social group is the graph, and a clique is an induced subgraph that is complete.



The illustration shows who is friends with who in a class of six schoolchildren. Amy, Bob and Cindy form a clique because each is friends with the other two. Cindy, Dede, Ed and Flo do not form a clique because Ed and Cindy are not friends, for example. Although Amy, Bob and Cindy form a clique, their clique is not maximal because Flo can be added into the friendship group and each possible pairing amongst the four is a friendship relation. The clique of Amy, Bob, Cindy and Flo is now maximal because no further child can be added to the four that would result in a larger clique. Notice that Dede and Ed, for example, also form a maximal clique as each member of that clique knows all the others, and no other child can be added that would result in a larger clique.

The following diagram illustrates the fact that any vertex induced subgraph of a complete graph forms a clique.



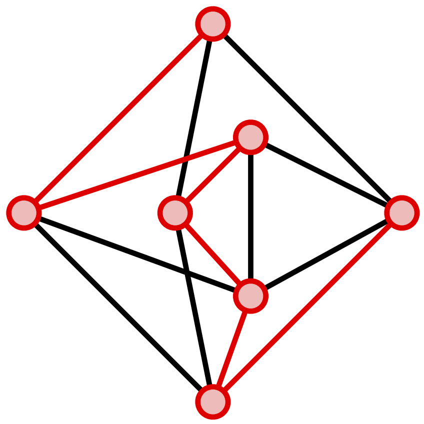
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow (0)$$

3.6 Some Other Pieces of Graph

Having dealt briefly with the subgraphs of a graph, we return to this lecture's main theme which was “patterns within a graph”. A crucial next step is to identify three entities known as path, trail and walk.

Definition : Path (of Distinct Vertices)

A path on n distinct vertices in a graph G is denoted, P_n. Discussed in section 1.3, a path will have two end vertices of degree at least 1 and thread its way through each of the remaining $(n - 2)$ vertices which will each be of degree at least 2. To emphasise; a path with at least one edge cannot be closed, that is, the first and last vertex cannot be the same vertex.



A path, P_7, (in red) on 7 vertices in graph G1169

Definition : Trail (of Distinct Edges)

A trail in a graph G is a (possibly empty) sequence of distinct edges in G such that any two consecutive edges in the sequence are incident to a common vertex. A trail allows repetitions of the same vertex whereas a path does not. If a non empty trail has only the first and last vertices equal it is termed a *cycle* or closed trail. If, in addition to having the same first and last vertex, it has other vertices occurring more than once it is termed a *circuit.*

Definition : Walk

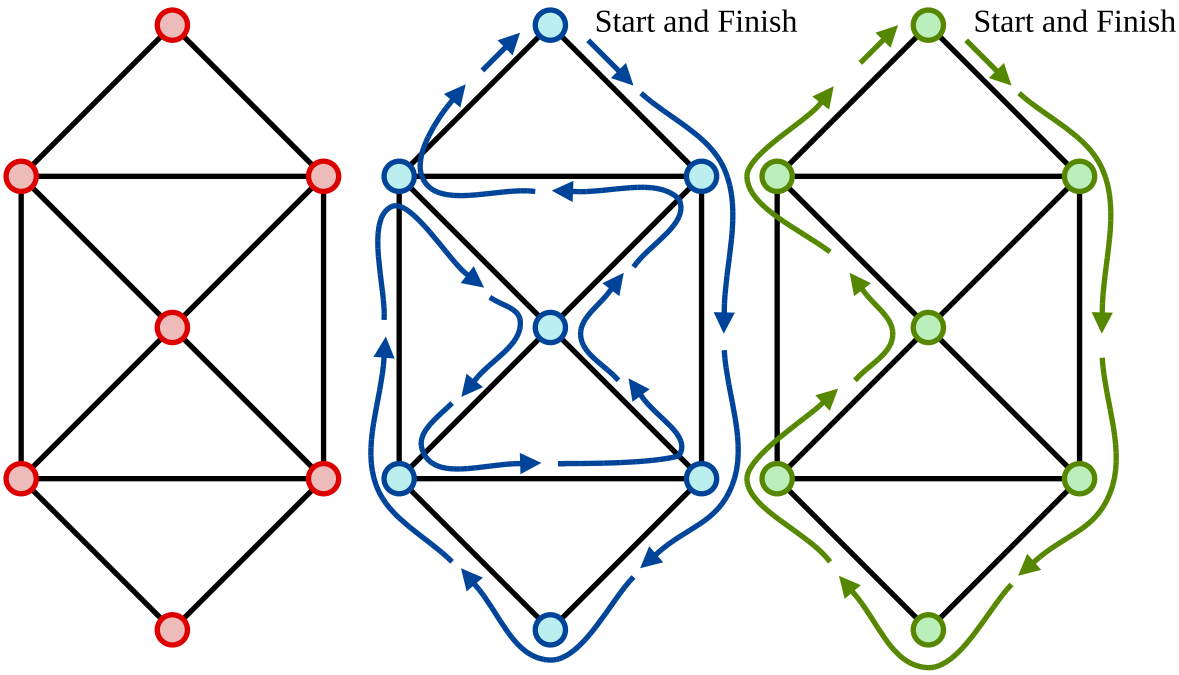
A walk on a graph G drops the requirement that the edges in the sequence of edges be distinct. A *closed walk* is a walk where the start and the finish are the same vertex.

3.7 The Eularian Edge Traveller

Given a connected graph, we can unleash upon it the Eularian Edge Traveller. This can be thought of as a spider that attempts to find a way to crawl along a closed trail that traverses each and every edge in the graph once and once only, starting and finishing at the same vertex. If such a closed trail can be found the graph is said to be *Eularian* and the trail said to be a *Eulerian trail***.**

For example, the graph below left (with red vertices) is Eularian and by way of demonstrating this a Eularian trail is shown in the central diagram (with blue vertices) threading its way through each edge once and once only.

Note that in this particular example there are other valid closed trails (even from the same start/finish vertex) that could have been used show that the graph is Eularian.



3.8 The Hamiltonian Vertex Visitor

Given a connected graph, we can unleash upon it the Hamiltonian Vertex Visitor. This can again be thought of as a spider. This time, however, it attempts to find a way to crawl in a cycle that passes through every vertex in the graph once and once only. If such a cycle can be found the graph is said to be Hamiltonian and the cycle is said to be a Hamiltonian cycle.

For example, the graph above left (with red vertices) is Hamiltonian. The third diagram in the illustration (with green vertices) shows this by exhibiting a closed trail that visits each vertex exactly once.

Note that in this particular example there are other valid cycles that could have been used to show that the graph is Hamiltonian.

3.9 Proofs

In this final section, we work our way through a sequence of "classic" elementary graph theory proofs to do with the connectivity of graphs.

Lemma 3.1 : On the Degree of the Vertices of an Eulerian Graph

If G is an Eulerian graph, then each vertex of G has even degree.

Proof

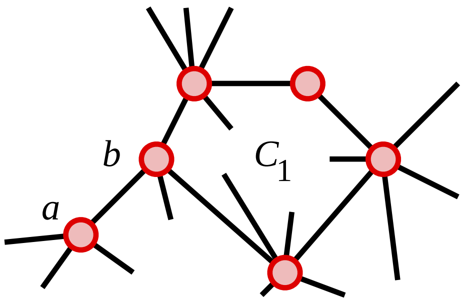
If G is Eulerian, then there is an Eulerian trail, that is, a closed trail that traverses each and every edge in the graph once and once only. Whenever this trail passes through a vertex there is a contribution of +2 to the degree of that vertex. Since each edge is used just once, the degree of each vertex is a sum of 2s. But that is precisely what an even number is. Thus, each vertex is of even degree. ☐

Lemma 3.2 : Eulerian Graph Decomposition

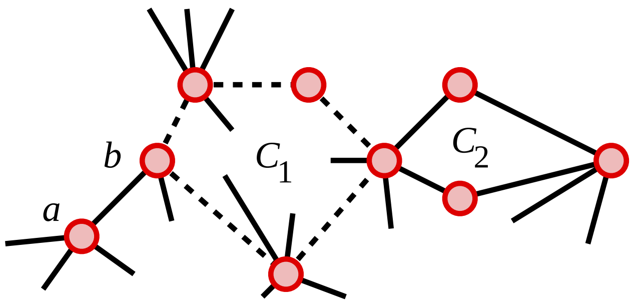
A graph that is Eulerian (and therefore connected) can be decomposed into distinct cycles, no two of which have an edge in common.

Proof

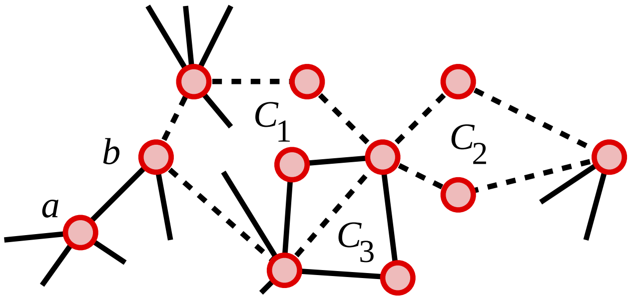
The first cycle is obtained by starting at any vertex a and traversing edges in an arbitrary manner, never repeating any edge. As the graph is Eulerian, each vertex is of even degree, by Lemma 3.1, which guarantees that whenever a vertex is entered, there is a way out. With only a finite number of vertices in G, eventually a vertex b is reached that has been previously encountered.



The edges of the closed trail between the two occurrences of the vertex b form a cycle, C_1 which is now removed from G, leaving a (possibly disconnected) graph H in which each vertex has even degree. If G is simply C_1 then we are done. If not then the procedure is repeated to find in H a cycle C_2 (which will not have any edges in common with C_1).



Removing the edges of C_2 from H leaves yet another graph in which each vertex has even degree, and which therefore contains a cycle C_3.



By continuing in this manner there will eventually be no edges left but, along the way, a set of distinct cycles C_1, C_2, C_3, ..., C_k have been obtained that include every edge of G, and no two of which have an edge in common. □

In Lemma 3.2 we began with the fact that a graph was Eularian to reason that each vertex was of even degree (by Lemma 3.1) and then deduced that the graph could be decomposed into distinct cycles. However, if we drop the initial requirement that the graph be Eularian, the argument employed in proving Lemma 3.2 will still apply from the starting point of having a graph with vertices all of even degree. Such a graph can still be decomposed into distinct cycles. Must it then be Eularian ? Our next lemma, Lemma 3.3, states that it is.

Lemma 3.3 : Graphs With Vertices All Even
If each vertex of a connected graph G has even degree, then G is Eulerian.

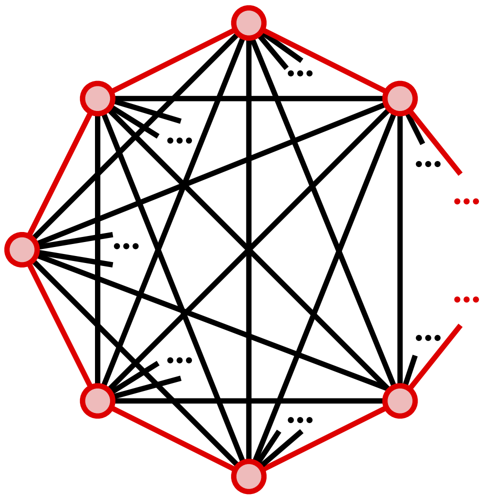
Proof
As just discussed, a connected graph in which each vertex is of even degree can be decomposed into distinct cycles. To now show this graph is Eularian the cycles or, rather, parts of the cycles, must be fitted together in some way to make an Eularian trail. Here is an algorithm that will do this; start at any vertex of the cycle C_1 and travel round C_1 until the vertex of another cycle is encountered such as, for example, C_2. Now pass along the edges of this cycle before resuming the journey around C_1, traversing other new cycles as they are encountered. At this juncture a closed trail that includes C_1 and (unless G is just C_1) at least one other cycle (since G is connected) has been traversed. If this trail includes all of the cycles C_1, C_2, C_3, ..., C_k then that is the required Eularian trail. If not, a journey is commenced round the new closed trail, traversing other cycles as they are met. There will always be at least one cycle to add to the previous closed trail because G is connected. The process is repeated in this manner until all the cycles have been traversed, whereupon the sought after Eularian trail is obtained. From this is deduced that G is Eularian. □

Theorem 3.1 : Eularian IFF
A connected graph is Eularian if and only if each vertex has even degree.

Proof
Lemma 3.1 and Lemma 3.2 in combination prove this theorem. □

Lemma 3.4 : K_n is Hamiltonian
The complete graph, K_n, is Hamiltonian for all values of n.

Proof
The complete graph K_n for any given value of n can be drawn with one vertex at each vertex of an n-gon as illustrated by the following diagram,



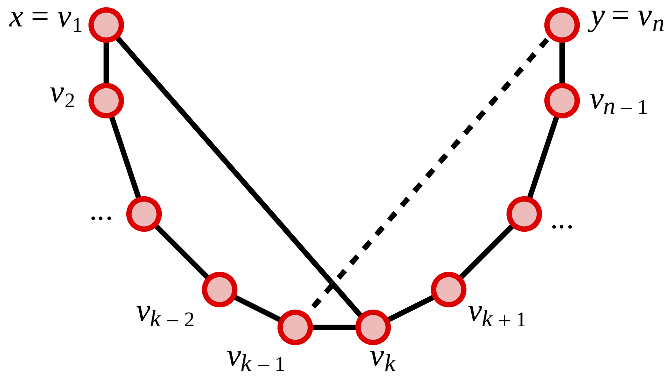
The perimeter of the diagram, highlighted in red, is the cycle subgraph C_n of K_n which is a cycle that includes every vertex.
This shows that K_n contains a Hamiltonian cycle for all values of n and so, by definition, is Hamiltonian. □

Theorem 3.2 A Test For Hamiltonianism (Ore, 1960)
If G is a simple graph with n ($\geqslant 3$) vertices, and if $deg(v) + deg(w) \geqslant n$ for each pair of non-adjacent vertices v and w, then G is Hamiltonian.

Proof (by contradiction)
Suppose, for a contradiction, that G is a simple graph with n ($\geqslant 3$) vertices, and with $deg(v) + deg(w) \geqslant n$ for each pair of non-adjacent vertices v and w but that G is not Hamiltonian. Pick any two vertices of G which are not already joined by an edge and add a new edge between them. Keep on doing this until a graph G_{Last} is obtained which, for the first time, does have a closed Hamiltonian path. (This event must eventually occur because the complete graph on n vertices acts as a "catching feature" because, by Lemma 3.4, the graph K_n is Hamiltonian).
Let G_{ButOne} be the graph immediately prior to G_{Last}, let $\{x, y\}$ be the edge added to G_{ButOne} to obtain G_{Last} and let $(z_1, z_2, ..., z_n, z_1)$ be a Hamiltonian cycle in G_{Last}. This must use the edge $\{x, y\}$ somewhere in the cycle because otherwise G_{ButOne} would have had a cycle (which would contradict the earlier requirement that it did not).
There are now two cases to consider;
Case 1 :
If $\{z_n, z_1\} = \{x, y\}$ then $(z_1, z_2, ..., z_n)$ is a non-closed Hamiltonian path in G_{ButOne}. Relabel the path's vertices $(v_1, v_2, ..., v_n)$ such that $x = v_1$ and $y = v_n$.

Case 2 :
If $\{z_n, z_1\} \neq \{x, y\}$ then there must be some r such that $1 \leqslant r < n$ and $\{x, y\} = \{z_r, z_{r+1}\}$ and $(z_{r+1}, \ldots, z_n, z_1, \ldots, z_r)$ is a non-closed Hamiltonian path in G_{ButOne}. Relabel the path's vertices $(v_1, v_2, \ldots, v_n)$ such that $x = v_1$ and $y = v_n$.

In both cases we emerge with a non-closed Hamiltonian path that is labelled $(v_1, v_2, \ldots, v_n)$ with $x = v_1$ and $y = v_n$ as illustrated below.



The focus now is upon the condition $deg(v) + deg(w) \geqslant n$.
The key observation is to note that for any vertex in the path that x is adjacent to, (say v_k where $2 \leqslant k \leqslant n$), y cannot be adjacent to the preceding vertex in the path (say v_{k-1} where $2 \leqslant k \leqslant n$) because the edge $\{x, y\}$ was specifically removed when constructing G_{ButOne} from G_{Last}.
Suppose for the sake of argument that y was adjacent to the preceding vertex as shown with a broken line. This cannot be the case because if it were then we would have the following Hamiltonian cycle,

$$x, v_2, \ldots, v_{k-2}, v_{k-1}, y, v_{n-1}, \ldots, v_{k+1}, v_k, x$$

Notice that this is only an issue if y is adjacent to the preceding vertex; for any other vertex a Hamiltonian cycle does not result.

Consequently, $deg(y) \leqslant n - 1 - deg(x)$ giving $deg(x) + deg(y) \leqslant n - 1$ which contradicts the earlier condition that $deg(v) + deg(w) \geqslant n$ for each pair of non-adjacent vertices v and w. □

The following teaching video from "Wrath of Math" goes through a proof of Ore's theorem along similar lines to that presented above;

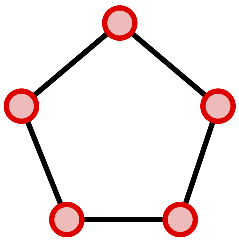
Teaching Video: http://www.NumberWonder.co.uk/v9119/3.mp4

3.10 Exercise

Marks Available : 60

Question 1

In section 1.3 a *cycle graph* was defined as being a graph that consists of a single cycle of vertices and edges and denoted C_n. The example of C_5 was given as;

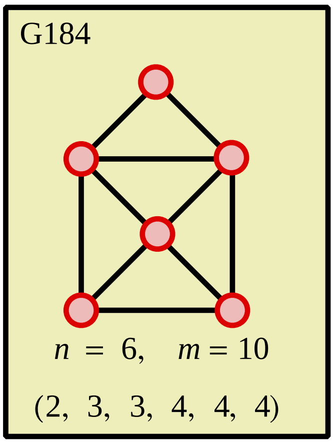


Charles claims that, "If every vertex of a simple graph G has degree 2, then G is a cycle". Exhibit a counterexample on six vertices that proves Charles is wrong.

[2 marks]

Question 2

(i) By first quoting a lemma or theorem, explain why G184 is not Eulerian.



[1 mark]

(ii) Although not Eulerian, the graph is semi-Eulerian.
This means that there exists a (non-closed) trail that includes every edge. Annotate the graph with such a trail.

[2 marks]

Question 3
Give an example of a connected graph on five vertices that is non-Eulerian (neither Eularian nor semi-Eularian) that contains a cycle.

[2 marks]

Question 4
Prove the following lemma;

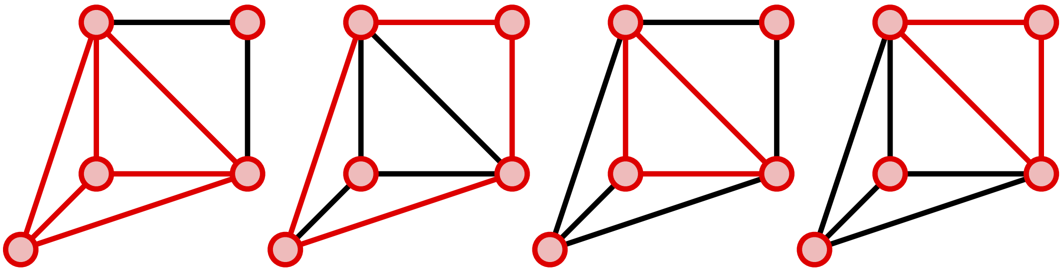
Lemma 3.5 : Must Contain A Cycle
Any finite, connected graph, with more than two vertices, in which every vertex is of degree of at least 2, must contain a cycle.

[4 marks]

Question 5

In each of the following four graphs, a part of the graph is highlighted in red.
Describe each of the highlighted configurations with one of the following;

- **(a)** "maximal clique"
- **(b)** "non-maximal clique"
- **(c)** "not a clique"



[4 marks]

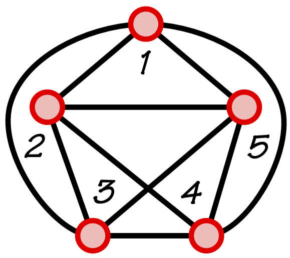
Question 6

The graph depicted below is of the complete graph, K_5.

It contains a trail that traverses each edge once and once only, with all vertices being encountered more than once.

Such a trail is, for example, 1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1

This is an example of an Euler circuit.



Explain why K_n does not contain an Euler circuit when n is even.

[3 marks]

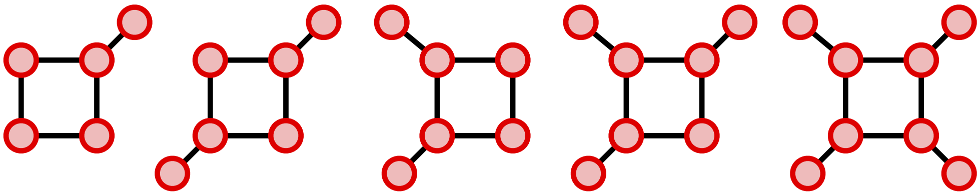
Question 7

Give an example on four vertices of a connected graph that has no Hamilton path.

[2 marks]

Question 8

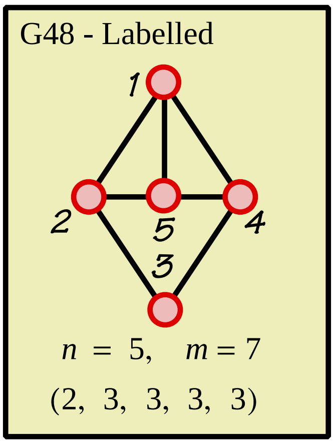
For each of these connected graphs state if they have a Hamiltonian path or not.



[3 marks]

Question 9

Show how Ore's Theorem correctly predicts that the following graph is Hamiltonian, and then annotate the graph to show such a cycle.



[3 marks]

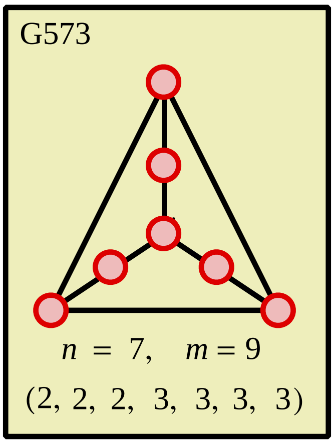
Question 10

Show that the cycle graph, C_5, is Hamiltonian in spite of Ore's Theorem not being satisfied.

[3 marks]

Question 11

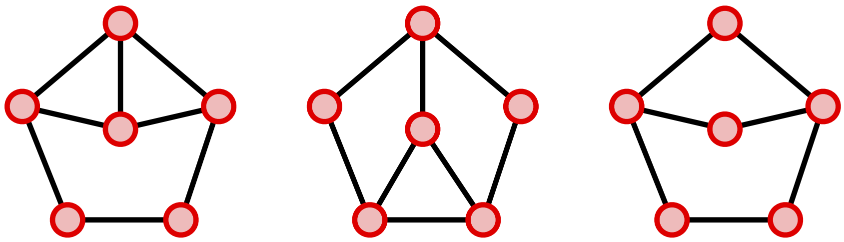
Explain why the following graph is not Hamiltonian.



[3 marks]

Question 12

A “Uniquely Hamiltonian Graph” is a graph possessing a single Hamiltonian cycle.
Determine which, if any, of the following are Uniquely Hamiltonian Graphs.
In each case, give a reason for your answer.



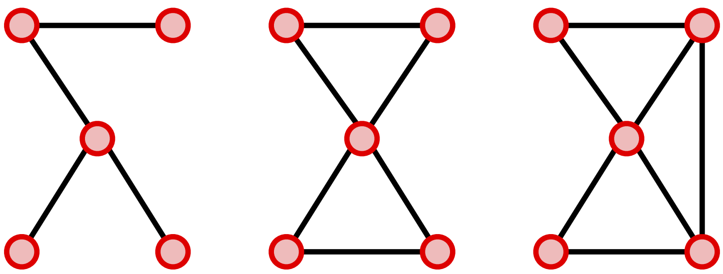
[3 marks]

Question 13

Determine if each of the following graphs is

(i) Eulerian

(ii) Hamiltonian



[3 marks]

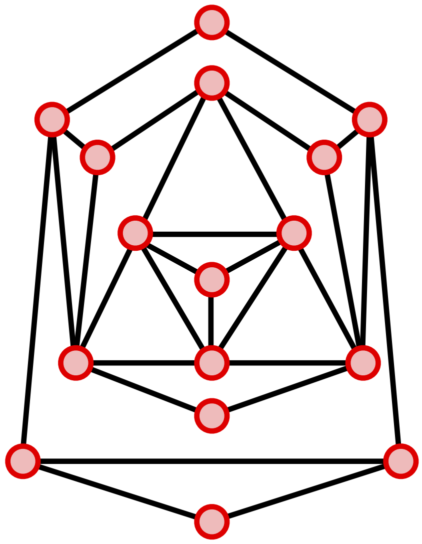
Question 14

Determine if the following graph is,

(i) Eulerian

(ii) Hamiltonian

Give a reason for each of your answers.

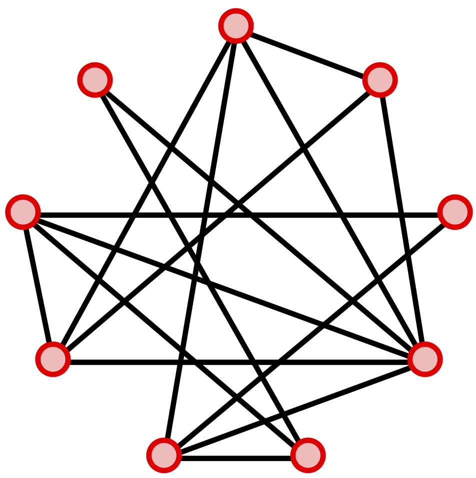


[4 marks]

Question 15

The graph represents friendships between a group of students where each vertex is a student and each edge is a friendship. Is it possible for the students to sit around a round table in such a way that every student sits between two friends ?

(With thanks to Oscar Levin for this question)



[4 marks]

Question 16

Prove the following corollary to Ore's Theorem,

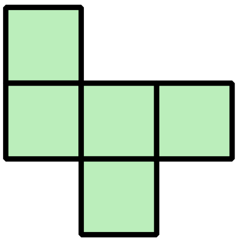
Corollary 3.1 : Dirac's Theorem

If G is a simple connected graph with n vertices, where $n \geqslant 3$, and $deg(v) \geqslant \dfrac{n}{2}$ for each vertex v, then G is Hamiltonian.

Hint : Use Ore's Theorem !

[4 marks]

Question 17

Consider the following pentomino, 

(i) Draw a graph G of this pentomino with five vertices and 4 edges.

[1 mark]

(ii) Draw a distinct pentomino whose graph is isomorphic to G.

[1 mark]

(iii) Draw two distinct pentominos whose graphs are not isomorphic to G, not isomorphic to each other, and not isomorphic to the path, P_5

[3 marks]

Question 18

In answering question 3, you will have found that the vertices of odd degree played a key role; one being the start of the (non-closed) trail and the other the finish. With that in mind the following lemma will not come as a surprise.

Lemma 3.6 : Semi-Eularian IFF

A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Prove lemma 3.4.

[5 marks]