2.8 Answers to 2.7 Exercise

Answer 1

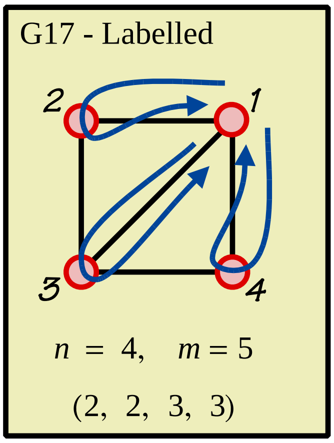
(i) $\mathbf{A}(\mathrm{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

[2 marks]

(ii) $\mathbf{A}^2(\mathrm{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$

[2 marks]

(iii) The three walks of length 2 between vertex *1* and itself are,



[1 mark]

Answer 2

$$\mathbf{A}^2 = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

(i) $b_{11} = (a_{11}\, a_{11}) + (a_{12}\, a_{21}) + (a_{13}\, a_{31}) + \dots + (a_{1n}\, a_{n1})$

As **A** is symmetric, $a_{ij} = a_{ji}$, so $b_{11} = (a_{11})^2 + (a_{12})^2 + (a_{13})^2 + \dots + (a_{1n})^2$

[1 mark]

(ii) Now, a_{11} is always zero, and each of the squares $(a_{1k})^2$ for $2 \leqslant k \leqslant n$ will be 1 when there is an edge between vertices v_1 and v_k, 0 otherwise. Thus b_{11} gives the degree of vertex v_1 and also the number of walks of length 2 between v_1 and itself. □

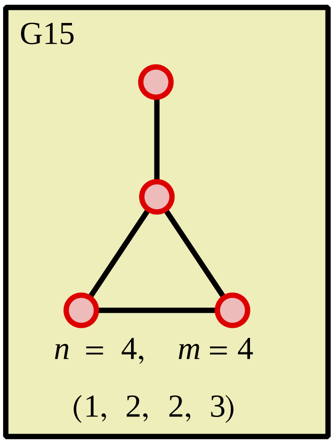
[2 marks]

(iii) $tr\left(\mathbf{A}^2\right)$ will give the sum of the degrees of all vertices in G which, by Theorem 1.2, The Handshaking Lemma, is twice the number of edges of G.

[2 marks]

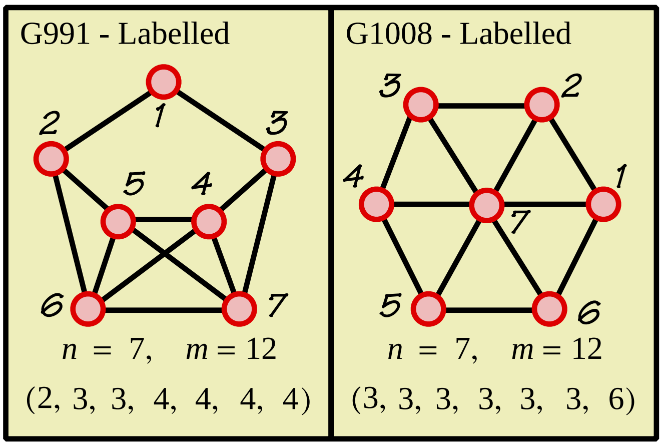
(iv) $\mathbf{H}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

The graph H has 4 vertices and degree sequence (1, 2, 2, 3).
This is enough to identify the graph as being G15



[2 marks]

Answer 3



(i)

$$\mathbf{A}(G991) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \mathbf{A}(G1008) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

[2 marks]

(ii)

$$\phi(G991) = \phi(G1008) = (x-1)^2(x+1)^2(x+2)(x^2-2x-6)$$

The two graphs are not isomorphic, yet their adjacency matrices have the same characteristic equation which, by definition, makes them cospectral.

[4 marks]

(iii)

$$\mathbf{A}^2(G991) = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 4 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 4 & 3 & 2 \\ 1 & 1 & 2 & 2 & 3 & 4 & 2 \\ 1 & 2 & 1 & 3 & 2 & 2 & 4 \end{pmatrix} \quad \mathbf{A}^2(G1008) = \begin{pmatrix} 3 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 2 & 1 & 2 & 2 \\ 2 & 1 & 3 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 3 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 6 \end{pmatrix}$$

$$\phi_2(G991) = \phi_2(G1008) = (x-4)(x-1)^4(x^2-16x+36)$$

Comment : The hope that the characteristic polynomials of the squares of the adjacency matrices might distinguish between the cospectral graphs is fundamentally flawed because,

“The matrix $\mathbf{A}^n$ has eigenvalue λ^n where λ is an eigenvalue of $\mathbf{A}$”

This statement may be proven using induction.

For a proof see Number Wonder's Matrix Algebra, Lecture 1, Question 5

https://www.NumberWonder.co.uk/Pages/Page9116.html

[4 marks]

Answer 4

Theorem 2.4 : Counting walks between vertices
Given a simple graph G with adjacency matrix $\mathbf{A}$, raising $\mathbf{A}$ to a positive integer power n gives a matrix where the entry a_{ij} gives the number of walks of length n between the vertices v_i and v_j

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Proof

To establish a basis for a proof by induction let $n = 1$ giving $\mathbf{A}^1 = \mathbf{A}$ which is the adjacency matrix for G in which entry $a_{ij}^{(1)}$ counts the number of walks of length 1 between v_i and v_j. As G is simple this count is either 1 if there is an edge between v_i and v_j or 0 if there is no edge.
The induction hypothesis is to assume true that when $n = k$ the number of walks of length k between v_i and v_j is the entry $a_{ij}^{(k)}$ in the matrix $\mathbf{A}^k$.
We can express a walk of length $k + 1$ between v_i and v_j of a walk of length k between v_i and v_u followed by a walk of length 1 from v_u to v_j.
In consequence, the number of walks of length $k + 1$ between v_i and v_j is the sum of all walks of length k from v_i to v_u multiplied by the number of ways to walk in one step from v_u to v_j. which is given by,

$$\sum_{r=1}^{n} a_{ir}^{(k)}\, a_{rj}$$

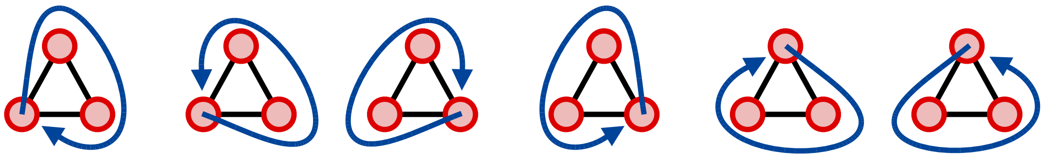
By the definition of matrix multiplication, this is the entry $a_{ij}^{(k+1)}$ in $\mathbf{A}^{k+1}$
Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction. □

[6 marks]

Answer 5

From Theorem 2.4 we know that, given a simple graph G with adjacency matrix $\mathbf{A}$, the elements on the diagonal of $\mathbf{A}^3$ (which are of the form $a_{ii}^{(3)}$) will be the walks of length 3 that start and finish at the same vertex. The only way that a walk of 3 steps can start and finish at the same vertex is if it is triangular. Let G be of order n.

The trace of $\mathbf{A}^3$ is $\sum_{i=1}^{n} a_{ii}^{(3)}$ which will be the sum of all triangular walks in G but with each counted six times as shown below.



Hence $tr\left(\mathbf{A}^3\right)$ gives six times the number of triangles in G. □

[4 marks]

Answer 6

Lemma 2.1 : Disconnected Detector

For a graph G of order n and adjacency matrix $\mathbf{A}$, calculate matrix $\mathbf{S}_n$ where,

$$\mathbf{S}_n = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots + \mathbf{A}^n$$

If there are any zeros in $\mathbf{S}_n$ then the graph is not connected.

Proof

If a graph is connected then the maximum length of a trail (a walk that does not traverse any edge more than once) is n. From theorem 2.4 we know that entries in the matrix $\mathbf{A}^k$ gives the number of walk of length k between all possible pairs of vertices in G. Thus a zero anywhere in the matrix $\mathbf{S}_n$ is telling us that between a pair of vertices in the graph there is no walk of length 1, 2, 3, ..., n. Thus there is a pair of vertices that have no way of connecting to each other.
In other words, the graph is disconnected. □

Note that there are other, more efficient, methods (especially as n becomes large) to determine if a graph is connected or not and, indeed, to determine the number of component parts.

[3 marks]

Answer 7

(i)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 3 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 4 & 5 & 1 & 1 & 1 \\ 1 & 4 & 2 & 4 & 1 & 1 & 1 \\ 1 & 5 & 4 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 2 & 4 & 4 \\ 0 & 1 & 1 & 1 & 4 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 & 3 & 2 \end{pmatrix}, \quad \mathbf{A}^4 = \begin{pmatrix} 3 & 2 & 4 & 5 & 1 & 1 & 1 \\ 2 & 12 & 7 & 7 & 7 & 2 & 2 \\ 4 & 7 & 8 & 7 & 6 & 2 & 2 \\ 5 & 7 & 7 & 14 & 4 & 6 & 6 \\ 1 & 7 & 6 & 4 & 13 & 6 & 6 \\ 1 & 2 & 2 & 6 & 6 & 7 & 6 \\ 1 & 2 & 2 & 6 & 6 & 6 & 7 \end{pmatrix}$$

[4 marks]

(ii) The Shimbel Matrix, **M**, is,

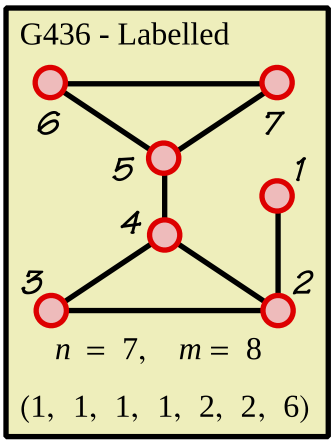
$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 1 & 1 & 2 & 3 & 3 \\ 2 & 1 & 2 & 1 & 2 & 3 & 3 \\ 2 & 1 & 1 & 2 & 1 & 2 & 2 \\ 3 & 2 & 2 & 1 & 2 & 1 & 1 \\ 4 & 3 & 3 & 2 & 1 & 2 & 1 \\ 4 & 3 & 3 & 2 & 1 & 1 & 2 \end{pmatrix}$$

[3 marks]

(iii) Diameter is 4

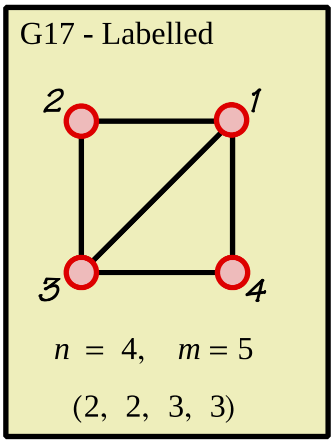
[1 mark]

Note that another way to answer this question would be to use the adjacency matrix to draw the graph and then simply study the graph to obtain **M**.



Answer 8

(i)



$$\mathbf{A}(\text{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^2(\text{G17}) = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A}^3(\text{G17}) = \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{pmatrix} \qquad \mathbf{A}^4(\text{G17}) = \begin{pmatrix} 15 & 9 & 14 & 9 \\ 9 & 10 & 9 & 10 \\ 14 & 9 & 15 & 9 \\ 9 & 10 & 9 & 10 \end{pmatrix}$$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(50 - 2\left(3 \times 2 + 2 \times 1 + 3 \times 2 + 2 \times 1 \right) - 10 \right)$$

$$= \frac{1}{8}\left(50 - 2 \times 16 - 10 \right)$$

$$= 1 \text{ simple 4-cycle}$$

[3 marks]

(ii)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{A}^2 = \begin{pmatrix} 3 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 3 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 4 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 3 \end{pmatrix}$$

$$2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right)$$ “*twice the sum of the triangularised degrees*”

$$= 2\left(6 + 12 + 6 + 12 + 6 + 6 + 6 + 12 + 6 + 6 + 6 \right.$$

$$= 2 \times 84$$

$$= 168$$

$$tr\left(\mathbf{A}^2\right) = 36 \text{ ``\textit{the sum of all degrees}''}$$

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 8 & 0 & 8 & 0 & 7 & 0 & 6 & 0 & 4 & 0 \\ 8 & 0 & 8 & 0 & 6 & 0 & 8 & 0 & 6 & 0 & 8 \\ 0 & 8 & 0 & 8 & 0 & 4 & 0 & 6 & 0 & 7 & 0 \\ 8 & 0 & 8 & 0 & 8 & 0 & 6 & 0 & 8 & 0 & 6 \\ 0 & 6 & 0 & 8 & 0 & 7 & 0 & 8 & 0 & 4 & 0 \\ 7 & 0 & 4 & 0 & 7 & 0 & 7 & 0 & 4 & 0 & 4 \\ 0 & 8 & 0 & 6 & 0 & 7 & 0 & 8 & 0 & 4 & 0 \\ 6 & 0 & 6 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \\ 0 & 6 & 0 & 8 & 0 & 4 & 0 & 8 & 0 & 7 & 0 \\ 4 & 0 & 7 & 0 & 4 & 0 & 4 & 0 & 7 & 0 & 7 \\ 0 & 8 & 0 & 6 & 0 & 4 & 0 & 8 & 0 & 7 & 0 \end{pmatrix}$$

$$\mathbf{A}^4 = \begin{pmatrix} 23 & 0 & 20 & 0 & 21 & 0 & 21 & 0 & 18 & 0 & 18 \\ 0 & 32 & 0 & 28 & 0 & 22 & 0 & 28 & 0 & 22 & 0 \\ 20 & 0 & 23 & 0 & 18 & 0 & 18 & 0 & 21 & 0 & 21 \\ 0 & 28 & 0 & 32 & 0 & 22 & 0 & 28 & 0 & 22 & 0 \\ 21 & 0 & 18 & 0 & 23 & 0 & 21 & 0 & 20 & 0 & 18 \\ 0 & 22 & 0 & 22 & 0 & 21 & 0 & 22 & 0 & 12 & 0 \\ 21 & 0 & 18 & 0 & 21 & 0 & 23 & 0 & 18 & 0 & 20 \\ 0 & 28 & 0 & 28 & 0 & 22 & 0 & 32 & 0 & 22 & 0 \\ 18 & 0 & 21 & 0 & 20 & 0 & 18 & 0 & 23 & 0 & 21 \\ 0 & 22 & 0 & 22 & 0 & 12 & 0 & 22 & 0 & 21 & 0 \\ 18 & 0 & 21 & 0 & 18 & 0 & 20 & 0 & 21 & 0 & 23 \end{pmatrix}$$

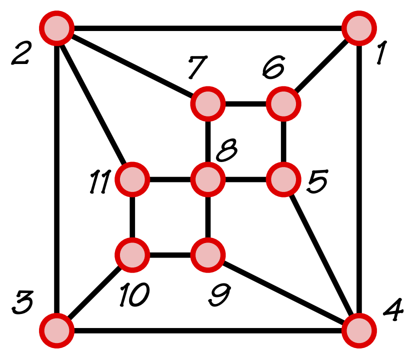
By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(276 - 168 - 36 \right)$$

$$= 9 \text{ simple 4-cycles}$$

From an inspection of the graph it can be seen that this is correct.



[4 marks]

Answer 9

(i)

$$\mathbf{A}(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{A}^2(K_4) = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

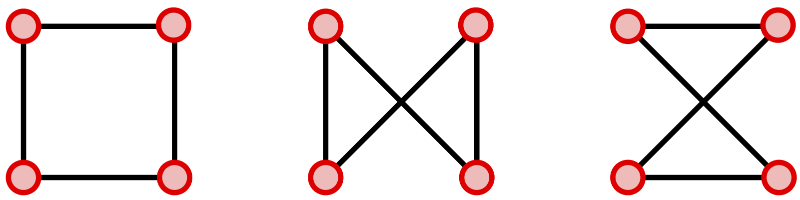
$$\mathbf{A}^3(K_4) = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix} \qquad \mathbf{A}^4(K_4) = \begin{pmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{pmatrix}$$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n} \left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(84 - 2 \times 24 - 12 \right)$$

$$= 3 \text{ simple 4-cycles}$$



[2 marks]

(ii)

$$\mathbf{A}(K_7) = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & \text{\Large 1} & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & \text{\Large 1} & & & & 0 & \\ & & & & & & 0 \end{pmatrix} \qquad \mathbf{A}^2(K_7) = \begin{pmatrix} 6 & & & & & & \\ & 6 & & & \text{\Large 5} & & \\ & & 6 & & & & \\ & & & 6 & & & \\ & \text{\Large 5} & & & 6 & & \\ & & & & & 6 & \\ & & & & & & 6 \end{pmatrix}$$

$$\mathbf{A}^3(K_7) = \begin{pmatrix} 30 & & & & & & \\ & 30 & & & \text{\Large 31} & & \\ & & 30 & & & & \\ & & & 30 & & & \\ & \text{\Large 31} & & & 30 & & \\ & & & & & 30 & \\ & & & & & & 30 \end{pmatrix} \qquad \mathbf{A}^4(K_7) = \begin{pmatrix} 186 & & & & & & \\ & 186 & & & \text{\Large 185} & & \\ & & 186 & & & & \\ & & & 186 & & & \\ & \text{\Large 185} & & & 186 & & \\ & & & & & 186 & \\ & & & & & & 186 \end{pmatrix}$$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n} \left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(1302 - 2 \times 210 - 42\right)$$

$$= 105 \text{ simple 4-cycles}$$

[3 marks]

(iii) This is about generalising parts (i) and (ii)

$$\mathbf{A}(K_n) = \begin{pmatrix} 0 & & & & & \\ & \cdots & & 1 & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & 1 & & & \cdots & \\ & & & & \cdots & \\ & & & & & 0 \end{pmatrix} \quad \mathbf{A}^2(K_n) = \begin{pmatrix} n-1 & & & & & \\ & \cdots & & n-2 & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & n-2 & & & \cdots & \\ & & & & \cdots & \\ & & & & & n-1 \end{pmatrix}$$

$$\mathbf{A}^3(K_n) = \begin{pmatrix} (n-1)(n-2) & & & & & \\ & \cdots & & (n+1)+(n-2)^2 & & \\ & & \cdots & & & \\ & & & \cdots & & \\ (n+1)+(n-2)^2 & & & & \cdots & \\ & & & & \cdots & \\ & & & & & (n-1)(n-2) \end{pmatrix}$$

$$\mathbf{A}^4(K_n) = \begin{pmatrix} (n-1)^2+(n-1)(n-2)^2 & & & & & \\ & \cdots & & 2(n-1)(n-2)+(n-2)^3 & & \\ & & \cdots & & & \\ & & & \cdots & & \\ 2(n-1)(n-2)+(n-2)^3 & & & & \cdots & \\ & & & & \cdots & \\ & & & & & (n-1)^2+(n-1)(n-2)^2 \end{pmatrix}$$

By Algorithm 2.1, the number of simple 4-cycles is given by,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

$$= \frac{1}{8}\left(n(n-1)^2 + n(n-1)(n-2)^2 - 2(n-1)(n-2)n - n(n-1)\right)$$

$$= \frac{n(n-1)}{8}\left((n-1) + (n-2)^2 - 2(n-2) - 1\right)$$

$$= \frac{n(n-1)}{8}\left(n - 1 + n^2 - 4n + 4 - 2n + 4 - 1\right)$$

$$= \frac{n(n-1)}{8}\left(n^2 - 5n + 6\right)$$

$$= \frac{n(n-1)(n-2)(n-3)}{8}$$

$$= \frac{n!}{8\,(n-4)!} \text{ simple 4-cycles}$$

There is a well known result that the number of simple m-cycles in K_n is given by $\frac{n!\,(k-1)!}{2\,k!\,(n-k)!}$ which for $k = 4$ matches our result.

[5 marks]

Answer 10

$$\mathbf{A}^2 = \begin{pmatrix} 3&0&1&1&0&0&1&1&1&1\\ 0&3&0&1&1&1&0&1&1&1\\ 1&0&3&0&1&1&1&0&1&1\\ 1&1&0&3&0&1&1&1&0&1\\ 0&1&1&0&3&1&1&1&1&0\\ 0&1&1&1&1&3&1&0&0&1\\ 1&0&1&1&1&1&3&1&0&0\\ 1&1&0&1&1&0&1&3&1&0\\ 1&1&1&0&1&0&0&1&3&1\\ 1&1&1&1&0&1&0&0&1&3 \end{pmatrix} \qquad \mathbf{A}^3 = \begin{pmatrix} 0&5&2&2&5&5&2&2&2&2\\ 5&0&5&2&2&2&5&2&2&2\\ 2&5&0&5&2&2&2&5&2&2\\ 2&2&5&0&5&2&2&2&5&2\\ 5&2&2&5&0&2&2&2&2&5\\ 5&2&2&2&2&0&2&5&5&2\\ 2&5&2&2&2&2&0&2&5&5\\ 2&2&5&2&2&5&2&0&2&5\\ 2&2&2&5&2&5&5&2&0&2\\ 2&2&2&2&5&2&5&5&2&0 \end{pmatrix}$$

$$\mathbf{A}^4 = \begin{pmatrix} 15&4&9&9&4&4&9&9&9&9\\ 4&15&4&9&9&9&4&9&9&9\\ 9&4&15&4&9&9&9&4&9&9\\ 9&9&4&15&4&9&9&9&4&9\\ 4&9&9&4&15&9&9&9&9&4\\ 4&9&9&9&9&15&9&4&4&9\\ 9&4&9&9&9&9&15&9&4&4\\ 9&9&4&9&9&4&9&15&9&4\\ 9&9&9&4&9&4&4&9&15&9\\ 9&9&9&9&4&9&4&4&9&15 \end{pmatrix}$$

$$\mathbf{A}^5 = \begin{pmatrix} 12&33&22&22&33&33&22&22&22&22\\ 33&12&33&22&22&22&33&22&22&22\\ 22&33&12&33&22&22&22&33&22&22\\ 22&22&33&12&33&22&22&22&33&22\\ 33&22&22&33&12&22&22&22&22&33\\ 33&22&22&22&22&12&22&33&33&22\\ 22&33&22&22&22&22&12&22&33&33\\ 22&22&33&22&22&33&22&12&22&33\\ 22&22&22&33&22&33&33&22&12&22\\ 22&22&22&22&33&22&33&33&22&12 \end{pmatrix}$$

By Algorithm 2.2, the number of simple 5-cycles is given by,

$$\frac{1}{10}\left(tr\left(\mathbf{A}^5\right) - 5\sum_{i=1}^{n}\left(\left(a_{ii}^{(3)}\right)\left(a_{ii}^{(2)} - 2\right)\right) - 5\,tr\left(\mathbf{A}^3\right)\right)$$

$$= \frac{1}{10}\left(120 - 0 - 0\right)$$

$$= 12 \text{ simple 5-cycles}$$

[5 marks]

Answer 11

(i) Each graph has 12 vertices. The "number of vertices" property has not detected any non-isomorphisms.

[1 mark]

(ii) Each graph has 18 edges. The "number of edges" property has not detected any non-isomorphisms.

[2 marks]

(iii) Each graph is 3-regular so the degree sequence of each is simply $(3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3)$. The "degree sequence" property has not detected any non-isomorphisms.

[2 marks]

(iv)

$$\mathbf{R} = \begin{pmatrix} 0&1&0&1&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&1&0&0&0&0&1&0 \\ 0&0&0&1&1&0&0&0&1&0&0&0 \\ 1&0&1&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&1&0&0&0&0&0&1 \\ 0&1&0&0&1&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&1&0&0&1&0&0&0&1&0 \\ 0&0&1&0&0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&1&1 \\ 0&1&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&1&0&0&0&1&0&0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0&1&0&0&0&0&1&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&1 \\ 0&1&0&1&0&0&0&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&0&0&1&0 \\ 0&0&0&1&0&1&0&0&0&1&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1&0&0&0&0 \\ 1&0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&1&0 \\ 0&0&1&0&1&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&1 \\ 0&1&0&0&0&1&0&0&0&0&1&0 \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} 0&1&0&0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&1&0&0 \\ 0&1&0&1&0&0&0&0&0&0&1&0 \\ 0&0&1&0&1&0&0&0&0&0&1&0 \\ 1&0&0&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&0&1 \\ 1&0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&0&1 \\ 0&1&0&0&0&0&0&0&1&0&1&0 \\ 0&0&1&1&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&1&0&1&0&0&0 \end{pmatrix}$$

[3 marks]

(v)

$$\mathbf{R}^3 = \begin{pmatrix} 0 & 5 & 1 & 6 & 2 & 0 & 6 & 1 & 1 & 2 & 2 & 1 \\ 5 & 0 & 2 & 1 & 1 & 5 & 1 & 2 & 2 & 1 & 5 & 2 \\ 1 & 2 & 0 & 5 & 5 & 1 & 2 & 1 & 5 & 1 & 2 & 2 \\ 6 & 1 & 5 & 0 & 0 & 2 & 1 & 6 & 2 & 2 & 1 & 1 \\ 2 & 1 & 5 & 0 & 2 & 5 & 1 & 1 & 1 & 2 & 2 & 5 \\ 0 & 5 & 1 & 2 & 5 & 2 & 1 & 1 & 2 & 2 & 1 & 5 \\ 6 & 1 & 2 & 1 & 1 & 1 & 0 & 6 & 5 & 1 & 2 & 1 \\ 1 & 2 & 1 & 6 & 1 & 1 & 6 & 0 & 2 & 1 & 5 & 1 \\ 1 & 2 & 5 & 2 & 1 & 2 & 5 & 2 & 0 & 5 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 5 & 0 & 5 & 5 \\ 2 & 5 & 2 & 1 & 2 & 1 & 2 & 5 & 1 & 5 & 0 & 1 \\ 1 & 2 & 2 & 1 & 5 & 5 & 1 & 1 & 1 & 5 & 1 & 2 \end{pmatrix}$$

$$tr\left(\mathbf{R}^3\right) \;=\; 6 \;\Leftrightarrow\; 1 \text{ triangle}$$

$$\mathbf{B}^3 = \begin{pmatrix} 2 & 5 & 0 & 1 & 1 & 2 & 5 & 5 & 1 & 2 & 2 & 1 \\ 5 & 0 & 5 & 1 & 3 & 1 & 2 & 1 & 3 & 0 & 1 & 5 \\ 0 & 5 & 0 & 6 & 0 & 3 & 1 & 2 & 1 & 6 & 2 & 1 \\ 1 & 1 & 6 & 0 & 6 & 1 & 1 & 1 & 2 & 1 & 5 & 2 \\ 1 & 3 & 0 & 6 & 0 & 5 & 0 & 2 & 1 & 6 & 1 & 2 \\ 2 & 1 & 3 & 1 & 5 & 0 & 5 & 1 & 3 & 0 & 1 & 5 \\ 5 & 2 & 1 & 1 & 0 & 5 & 2 & 5 & 1 & 2 & 2 & 1 \\ 5 & 1 & 2 & 1 & 2 & 1 & 5 & 2 & 5 & 0 & 0 & 3 \\ 1 & 3 & 1 & 2 & 1 & 3 & 1 & 5 & 0 & 5 & 5 & 0 \\ 2 & 0 & 6 & 1 & 6 & 0 & 2 & 0 & 5 & 0 & 2 & 3 \\ 2 & 1 & 2 & 5 & 2 & 1 & 2 & 0 & 5 & 2 & 0 & 5 \\ 1 & 5 & 1 & 2 & 1 & 5 & 1 & 3 & 0 & 3 & 5 & 0 \end{pmatrix}$$

$$tr\left(\mathbf{B}^3\right) \;=\; 6 \;\Leftrightarrow\; 1 \text{ triangle}$$

$$\mathbf{G}^3 = \begin{pmatrix} 0 & 5 & 1 & 1 & 5 & 2 & 5 & 1 & 3 & 0 & 3 & 1 \\ 5 & 0 & 6 & 3 & 1 & 1 & 0 & 2 & 0 & 6 & 1 & 2 \\ 1 & 6 & 2 & 5 & 2 & 1 & 1 & 0 & 2 & 1 & 6 & 0 \\ 1 & 3 & 5 & 2 & 5 & 0 & 1 & 1 & 1 & 2 & 5 & 1 \\ 5 & 1 & 2 & 5 & 0 & 5 & 2 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 5 & 0 & 1 & 7 & 0 & 2 & 1 & 7 \\ 5 & 0 & 1 & 1 & 2 & 1 & 0 & 7 & 0 & 3 & 0 & 7 \\ 1 & 2 & 0 & 1 & 1 & 7 & 7 & 0 & 7 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 2 & 0 & 0 & 7 & 0 & 5 & 0 & 7 \\ 0 & 6 & 1 & 2 & 2 & 2 & 3 & 0 & 5 & 0 & 6 & 0 \\ 3 & 1 & 6 & 5 & 1 & 1 & 0 & 1 & 0 & 6 & 2 & 1 \\ 1 & 2 & 0 & 1 & 1 & 7 & 7 & 0 & 7 & 0 & 1 & 0 \end{pmatrix}$$

$$tr\left(\mathbf{G}^3\right) \;=\; 6 \;\Leftrightarrow\; 1 \text{ triangle}$$

The “triangle count” property has not detected any non-isomorphisms.

[3 marks]

(vi)

The characteristic equations are:

For **R**:

$$(x + 2)(x^4 + x^3 - 4x^2 - x + 2)(x^5 - 7x^3 + x^2 + 11x - 4)$$

For **B**:

$$(x - 3)(x + 1)(x^2 - 2)(x^3 + x^2 - 2x - 1)(x^5 + x^4 - 8x^3 - 3x^2 + 16x - 6)$$

For **G**:

$$(x - 3)\, x^2 (x^9 + 3x^8 - 9x^7 - 29x^6 + 22x^5 + 82x^4 - 17x^3 - 77x^2 + 3x + 13)$$

These show that none of the graphs are isomorphic to any of the others.