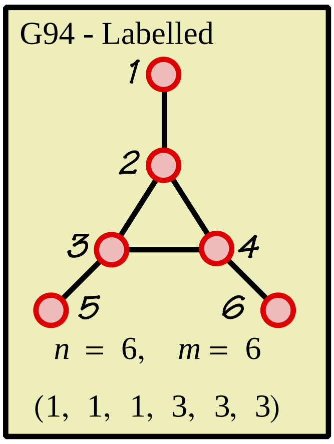
Lecture 2

Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

2.1 The Adjacency Matrix

By definition, the vertices v and w of a graph are *adjacent* vertices if they are joined by an edge, e. If G is a graph with n vertices (labelled 1, 2, 3, ..., n) then the *adjacency matrix* $\mathbf{A}(G)$ of G is the $n \times n$ square matrix in which the entry a_{ij} is the number of edges joining the vertices i and j. For a graph that is *simple* the entry can only be 0 or 1. Below, as an example, is the graph G94 which has been labelled and next to it is given its adjacency matrix. The top row of this adjacency matrix shows that the vertex labelled *1* is connected only to the vertex labelled *2*, the second row shows that the vertex labelled *2* has a direct connection to the vertices labelled *1*, *3* and *4*, and subsequent rows show how the remaining vertices connect.



$$\mathbf{A}(\text{G94}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In general the adjacency matrix of a simple graph will be symmetric and have a leading diagonal of all zeros. Interest in adjacency matrices centres around identifying properties of graphs that are captured by them. For example, the trace of a square matrix is the sum of its diagonal entries and denoted by $tr(\mathbf{A})$. It turns out that $tr\left(\mathbf{A}^2\right)$ gives twice the number of edges of the associated graph and $tr\left(\mathbf{A}^3\right)$ gives six times the number of triangles, as illustrated below.

$$\mathbf{A}^2(\text{G94}) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow tr\left(\mathbf{A}^2(\text{G94})\right) = 12 \quad \therefore 6 \textit{ edges}$$

$$\mathbf{A}^3(\text{G94}) = \begin{pmatrix} 0 & 3 & 1 & 1 & 1 & 1 \\ 3 & 2 & 5 & 5 & 1 & 1 \\ 1 & 5 & 2 & 5 & 3 & 1 \\ 1 & 5 & 5 & 2 & 1 & 3 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 & 1 & 0 \end{pmatrix} \Rightarrow tr\left(\mathbf{A}^3(\text{G94})\right) = 6 \quad \therefore 1 \textit{ triangle}$$

Performing calculations and manipulations on large matrices is tedious by hand and more reliably done using computer software.

Let $\phi(X,\ x)$ denote the characteristic polynomial of $\mathbf{A}(x)$.

For the adjacency matrix $\mathbf{A}$(G94) software gives its characteristic equation as,

$$\phi(X,\ x) = \left(x^2 - 2x - 1\right)\left(x^2 + x - 1\right)^2$$

The spectrum of a matrix is the list of its eigenvalues together with their multiplicities. For $\mathbf{A}$(G94) the spectrum is,

$$\left\{1 \pm \sqrt{2},\ \frac{-1 \pm \sqrt{5}}{2}^{(2)}\right\}$$

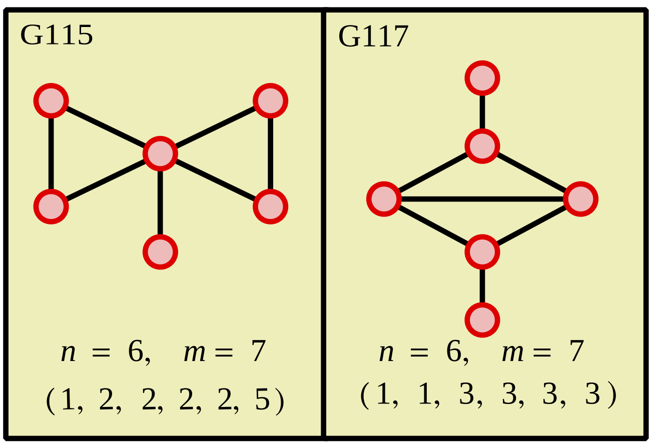
where the superscripts give the multiplicities that are greater than one.

Let $\alpha = 1 \pm \sqrt{2}$, and $\beta = \dfrac{-1 \pm \sqrt{5}}{2}$

The six eigenvectors of $\mathbf{A}$(G94) are then,

$$v(\lambda = \alpha) = \begin{pmatrix} 1 \\ \alpha \\ \alpha \\ \alpha \\ 1 \\ 1 \end{pmatrix} \qquad v(\lambda = \beta) = \begin{pmatrix} -1 \\ -\beta \\ 0 \\ \beta \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} -1 \\ -\beta \\ \beta \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Of ongoing interest is determining if two graphs are isomorphic from the mathematics associated with them. If two graphs, G and H are isomorphic then, although they have different adjacency matrices $\mathbf{A}(G)$ and $\mathbf{A}(H)$, they will have the same characteristic equation and spectrum. However, this cannot be used the other way round; cospectral graphs are not isomorphic, yet have the same characteristic equation and spectrum.



G115 and G117 provide an example of cospectral graphs. They are clearly not isomorphic as G115 has one vertex of degree 1 (a leaf) whereas G117 has two. Yet they both have the same characteristic equation.

$$\phi(\text{G115},\ x) = \phi(\text{G117},\ x) = \left(x - 1\right)\left(x + 1\right)^2\left(x^3 - x^2 - 5x + 1\right)$$

An objective of this lecture is to show that, in spite of the cospectral set back, it is possible to determine if two graphs are isomorphic from their adjacency matrices. However, to do so requires the prior development of a few ideas and it is to these we now attend.

2.2 Transposed Matrices

Here is a brief reminder of what the transpose of a matrix is.

The Transpose of an $n \times n$ Matrix

Given the matrix $\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ the transpose of matrix **M**

is denoted $\mathbf{M}^{\mathrm{T}}$ and is formed by an interchange of rows and columns.

Thus, $$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

2.3 Symmetric Matrices

A matrix, **M**, is symmetric if $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$. Such matrices are readily recognised for their elements are symmetric with respect to the leading diagonal. The adjacency matrix of a graph is symmetric as are powers of that matrix which means that the properties of such matrices will be of importance.

2.4 Permutation Matrices

A permutation matrix, **P**, is an $n \times n$ square matrix such that each row and each column contains a single element equal to 1, the remaining elements being 0. Consider the following calculation which involves a permutation matrix,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} d \\ b \\ a \\ e \\ c \end{pmatrix}$$

This permutation matrix has permutated the letters a, b, c, d, e as shown below,

$$\begin{pmatrix} a & b & c & d & e \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ d & b & a & e & c \end{pmatrix}$$

In cycle notation this permutation could be written $(adec)(b)$ or just $(adec)$.

A permutation matrix can permutate an entire matrix in two different ways as the following two calculations illustrate,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & f & k & p & u \\ b & g & l & q & v \\ c & h & m & r & w \\ d & i & n & s & x \\ e & j & o & t & y \end{pmatrix} = \begin{pmatrix} d & i & n & s & x \\ b & g & l & q & v \\ a & f & k & p & u \\ e & j & o & t & y \\ c & h & m & r & w \end{pmatrix}$$

$$\begin{pmatrix} a & f & k & p & u \\ b & g & l & q & v \\ c & h & m & r & w \\ d & i & n & s & x \\ e & j & o & t & y \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} p & f & a & u & k \\ q & g & b & v & l \\ r & h & c & w & m \\ s & i & d & x & n \\ t & j & e & y & o \end{pmatrix}$$

In the first calculation it is the rows of the lettered matrix that have been permutated. Notice that in the second of these calculations the transpose of the permutation matrix has been used. In this calculation it is the columns of the lettered matrix that have been permutated.

All of this illustrates the next theorem.

Theorem 2.1 : Permutating Rows and Columns

Given a square matrix, **S**, and a permutation matrix, **P**, multiplying by **P** on the left permutates the rows of **S**, whilst multiplying by $\mathbf{P}^{\mathrm{T}}$ on the right permutates the columns.

PS permutates rows of **S**

$\mathbf{SP}^{\mathrm{T}}$ permutates columns of **S**

In general there are $n!$ permutation matrices of size $n \times n$. Of the six 3×3 permutation matrices three are elementary permutation matrices that swap just two rows or two columns. Here are those six matrices. Those that are elementary are highlighted in red, and the (left multiplying) permutating effect they would have on the column vector $(a, b, c)^{\mathrm{T}}$ is given immediately underneath each.

$$\underset{(a)\,(b)\,(c)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \underset{(bc)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}, \underset{(ac)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}, \underset{(ab)}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \underset{(acb)}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}, \underset{(abc)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

In general, of the $n!$ permutation matrices of dimension $n \times n$ the number that are elementary is given by the triangular number T_{n-1} where,

$$T_n = \frac{n(n+1)}{2}$$

Clearly, any permutation matrix raised to a sufficient power will yield the identity matrix, **I**. The following demonstrates this fact for each of the six 3 × 3 permutation matrices;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^3$$

$$(a)\,(b)\,(c) \quad = \quad (bc)^2 \quad = \quad (ac)^2 \quad = \quad (ab)^2 \quad = \quad (acb)^3 \quad = \quad (abc)^3$$

In general, a non-elementary permutation matrix can be decomposed into a product of elementary permutation matrices. Again, this is a fact that can be demonstrated for the 3 × 3s, although there is a catch, shortly to be explained;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2$$

$$(a)\,(b)\,(c) \quad = \quad (bc)^2 \quad = \quad (ac)^2 \quad = \quad (ab)^2$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(abc) \quad = \quad (ac) \qquad (ab)$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(acb) \quad = \quad (ab) \qquad (ac)$$

The astute reader looking at the above three matrix equations may be wondering why the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ can be associated with different permutations. This is a much glossed over issue that an internet search will do little to explain. When this matrix acts on $(a, b, c)^{\mathrm{T}}$ it represents the permutation (bc). However when it acts upon $(b, a, c)^{\mathrm{T}}$ it represents (ac) and when it acts upon $(c, b, a)^{\mathrm{T}}$ it represents the permutation (ab). Marrying up a matrix with the permutation it represents is not as straight forward as one might initially have expected; it depends upon how preceding matrices have permutated the rows (or columns).

Permutations and their manipulation in cycle notation are covered in the Number Wonder undergraduate lectures, Group Theory II.
https://www.NumberWonder.co.uk/Pages/Page9110.html

2.5 Orthogonality

A matrix, $\mathbf{Q}$, is described as being orthogonal if it is a real square matrix and has the property that $\mathbf{QQ}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ where $\mathbf{Q}^{\mathrm{T}}$ is the transpose of $\mathbf{Q}$ and $\mathbf{I}$ is the identity matrix. This immediately leads to an equivalent characterization of orthogonality; $\mathbf{Q}$ is orthogonal if its transpose is equal to its inverse, if $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}$

Definition : An Orthogonal Matrix

A real matrix is orthogonal iff it is invertible and its inverse is its transpose.

The interest in orthogonality stems from the fact that permutation matrices have this property. For example, here the permutation introduced at the start of section 2.4 is multiplied by its transpose and the result, as claimed, is indeed the 5×5 identity matrix,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 2.2 : Permutation Matrices are Orthogonal

The product of a permutation matrix and its transpose gives the identity matrix.

That is, $$\mathbf{P}\,\mathbf{P}^{\mathrm{T}} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$$

If follows that permutation matrices are orthogonal.

Proof

Clearly the $n \times n$ identity matrix, $\mathbf{I}$, is orthogonal for all positive integer values of n. If any two rows in $\mathbf{I}$ or any two columns in $\mathbf{I}$ are swapped the result is an elementary permutation matrix which retains the property of being orthogonal because it is still symmetric and still coincides with its inverse. This proves the theorem in the case of the elementary permutation matrices.

Any non-elementary permutation matrix, $\mathbf{P}$, can be decomposed into a product of elementary permutation matrices, $\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_k$ and we now argue as follows;

$$\mathbf{P}^{-1} = \left(\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_k\right)^{-1} = \mathbf{P}_k^{-1} \ldots \mathbf{P}_2^{-1}\,\mathbf{P}_1^{-1} = \mathbf{P}_k^{\mathrm{T}} \ldots \mathbf{P}_2^{\mathrm{T}}\,\mathbf{P}_1^{\mathrm{T}} = \left(\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_k\right)^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}}$$

which completes the proof.

□

2.6 Isomorphism

For two graphs to be isomorphic there are many properties that must be common. They must have the same number of vertices, or edges, or spectrum, for example. However, non isomorphic graphs can have the same number of edges, for example, and it was shown previously that non isomorphic graphs can even have the same spectrum (cospectral graphs). Determining if two graphs are isomorphic or not can be a frustrating business. As all of the structure of a graph is captured by its adjacency matrix it is in principle possible to determine if two graphs are isomorphic. Write down the adjacency matrix of each, and then search for a permutation matrix for which Theorem 2.3, stated next, holds.

Theorem 2.3 : Isomorphism via Adjacency Matrices
Let G and H be graphs on the same vertex set and with adjacency matrices $\mathbf{A}(G)$ and $\mathbf{A}(H)$ respectively. Then G and H are isomorphic if and only if there is a permutation matrix $\mathbf{P}$ such that,

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}(G)\,\mathbf{P} = \mathbf{A}(H)$$

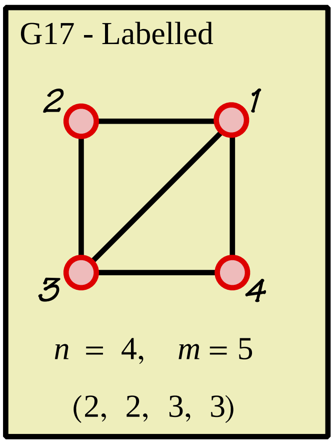
Being able to state Theorem 2.3 and to have developed the mathematics to understand what it is saying has been the goal of this lecture. However, in some respects it is a damp squib. This is because in practice, theorem 2.3 is of limited use in the general case; for a graph with n vertices, there are $n!$ candidates to be the sought after permutation matrix.

2.7 Exercises

Marks Available : 80

Question 1

(i) Write down the adjacency matrix $\mathbf{A}$(G17) for G17, shown below,



[2 marks]

(ii) By hand, write down the matrix $\mathbf{A}^2$(G17) which will give the number of walks of length 2 between the various vertices.

[2 marks]

(iii) Verify from the graph of G17 that there are three walks of length 2 between vertex 1 and itself.

[1 mark]

Question 2

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$ be the adjacency matrix of a labelled graph G

where the entry a_{ij} is 1 if there is an edge between vertex v_i and v_j, and 0 otherwise.

(i) Write down an expression for the top left entry of $\mathbf{A}^2$

[1 mark]

(ii) Explain why this counts the number of walks of length 2 between vertex 1 and itself.

[2 marks]

(iii) Building on your part (ii) answer, explain why $tr\left(\mathbf{A}^2\right)$ gives twice the number of edges of the associated graph.

[2 marks]

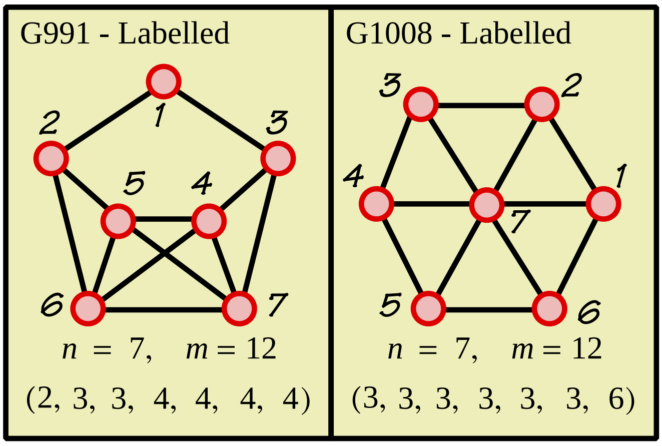
(iv) A graph, H, has adjacency matrix $\mathbf{H}$ such that $\mathbf{H}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

Draw the graph of H.

[2 marks]

Question 3

The graphs G991 and G1008, shown below, are clearly not isomorphic as they have different degree sequences.



(i) Write down the adjacency matrix for each graph.

[2 marks]

(ii) Use software to find the characteristic polynomials for G991 and G1008 and hence deduce that these two graphs are cospectral.

[4 marks]

(iii) Find the characteristic polynomial for the square of each adjacency matrix and comment.

[4 marks]

Question 4

Theorem 2.4 : Counting walks between vertices
Given a simple graph G with adjacency matrix $\mathbf{A}$, raising $\mathbf{A}$ to the power n gives a matrix where the entry a_{ij} gives the number of walks of length n between the vertices v_i and v_j

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Write out a proof by induction for Theorem 2.4

[6 marks]

Question 5

For a simple graph G with adjacency matrix $\mathbf{A}$, explain why $tr\left(\mathbf{A}^3\right)$ gives six times the number of triangles in G.
You may quote Theorem 2.4 (from question 4) as a part of your explanation.

[4 marks]

Question 6

Lemma 2.1 : Disconnected Detector
For a graph G of order n and adjacency matrix $\mathbf{A}$, calculate matrix $\mathbf{S}_n$ where,

$$\mathbf{S}_n = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots + \mathbf{A}^n$$

If there are any zeros in $\mathbf{S}_n$ then the graph is not connected.

Give a short proof of Lemma 2.1
You may quote Theorem 2.4 (from question 4)as a part of your explanation.

[3 marks]

Question 7

Lemma 2.2 Shortest Path Between a Vertex Pair
For a graph G of order n and adjacency matrix $\mathbf{A}$, calculate matrix $\mathbf{S}_k$ where,

$$\mathbf{S}_k = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots + \mathbf{A}^k, \quad k \leqslant n$$

The entry in row i and column j of matrix $\mathbf{S}_k$ tallies the number of ways to get from vertex v_i to vertex v_j in k steps or less. (A step is the traversal of an edge). To find the shortest number of steps between v_i and v_j begin to calculate the partial sums $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, ..., $\mathbf{S}_n$. Then, the first value of k for which the entry in row i and column j of matrix $\mathbf{S}_k$ is non-zero is the shortest number of steps.

A graph, G, has adjacency matrix, $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

(i) As necessary, use software to write down $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$

[4 marks]

(ii) Use your part (i) answer to write out the Shimbel Matrix, $\mathbf{M}$, for G where the entry row i and column j of matrix $\mathbf{M}$ is the least number of steps between the vertices v_i and v_j

[3 marks]

(iii) The diameter of a graph is the shortest path between the most distant distant minimum vertices. This is the largest value in the Shimbel Matrix. State the diameter of G.

[1 mark]

Question 8

A “simple” 4-cycle is a closed walk around four distinct vertices of the form

v_a - v_b - v_c - v_d - v_a

This excludes walks of the form v_a - v_b - v_a - v_b - v_a

and v_a - v_b - v_a - v_d - v_a

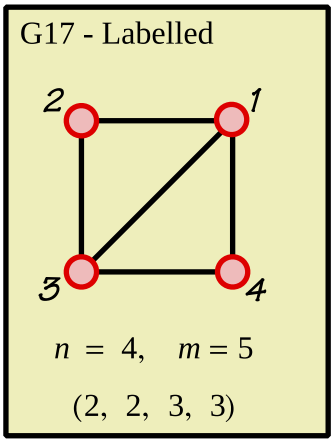
and v_a - v_b - v_c - v_b - v_a

Algorithm 2.1 : Counting Simple 4-Cycles

For a graph G with adjacency matrix $\mathbf{A}$, the number of proper 4-cycles is,

$$\frac{1}{8}\left(tr\left(\mathbf{A}^4\right) - 2\sum_{i=1}^{n}\left(\left(a_{ii}^{(2)}\right)\left(a_{ii}^{(2)} - 1\right)\right) - tr\left(\mathbf{A}^2\right)\right)$$

(i) From question 1, for the graph of G17 we know that,



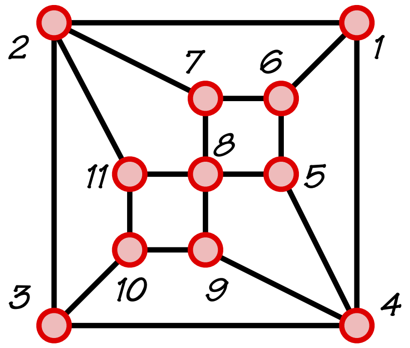
$$\mathbf{A}(\text{G17}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^2(\text{G17}) = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Show that Algorithm 2.1 correctly finds a single simple 4-cycle in G17.

[3 marks]

(ii) Use Algorithm 2.1 to find the number of simple 4-cycles in the graph G877 which is shown below. This graph is sufficiently small so that you can see what the correct answer should be !



[4 marks]

Question 9

Using matrix methods, how many simple 4-cycles are there in the graph,

(i) K_4

[2 marks]

(ii) K_7

[3 marks]

(iii) K_n

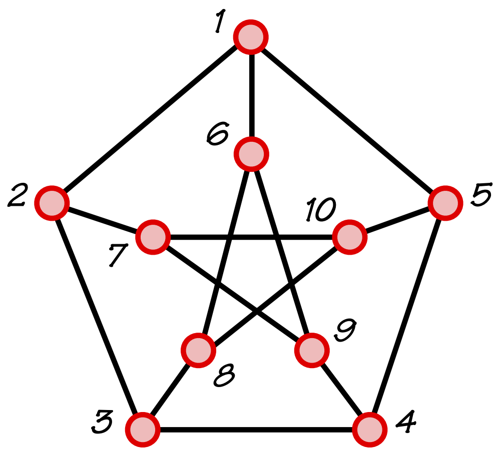
[5 marks]

Question 10

Algorithm 2.2 : Counting Simple 5-Cycles
For a graph G with adjacency matrix $\mathbf{A}$, the number of proper 5-cycles is,

$$\frac{1}{10}\left(tr\left(\mathbf{A}^5\right) - 5\sum_{i=1}^{n}\left(\left(a_{ii}^{(3)}\right)\left(a_{ii}^{(2)} - 2\right)\right) - 5\,tr\left(\mathbf{A}^3\right)\right)$$

Shown is the graph and adjacency matrix for the Petersen graph.



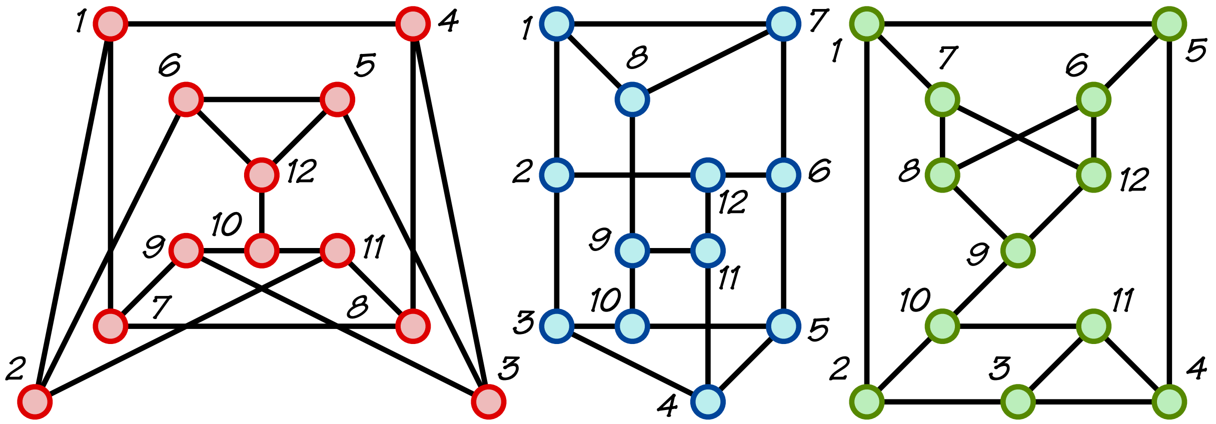
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Use Algorithm 2.1 to find the number of simple 5-cycles in the Petersen graph.

[5 marks]

Question 11

The purpose of this question is to investigate the isomorphisms (if any) between the three (labelled) graphs presented below.



(i) State the number of vertices of each graph.
Does this identify if any of the three are non-isomorphic to the others ?

[1 mark]

(ii) State the number of edges of each graph.
Does this identify if any of the three are non-isomorphic to the others ?

[1 mark]

(iii) State the degree sequence of each graph.
Does this identify if any of the three are non-isomorphic to the others ?

[1 mark]

(iv) Construct the adjacency matrix for each graph.

[3 marks]

(v) Use computer software to cube each graph's adjacency matrix.
For each of these matrices calculate $tr\left(\mathbf{A}^3\right)$.
Hence state the number of triangles in each graph.
Does this identify if any of the three are non-isomporphic to the others ?

[3 marks]

(vi) Use software to determine the characteristic equation for each graph's adjacency matrix and hence find the associated spectrum.
Does this identify if any of the three are non-isomporphic to the others ?

[6 marks]