1.8 Answers to Exercise 1.7

Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

Answer 1

(i) The degree sequence for K_4 is (3, 3, 3, 3)

(ii) No edges cross on a planar graph.
Graph B is not planar.

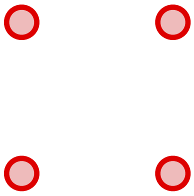
(iii) At degree of each vertex is three and so K_4 is 3-regular.

(iv) Graph C has four faces (don't forget the external face).

(v) K_4 could represent a tetrahedron.

[5 marks]

Answer 2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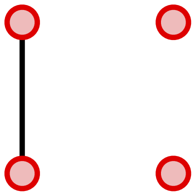
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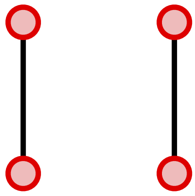
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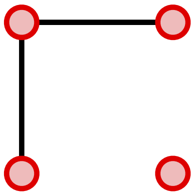
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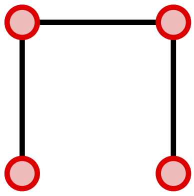
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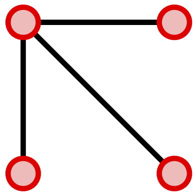
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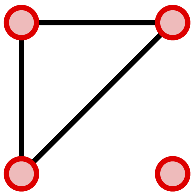
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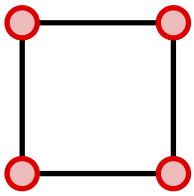
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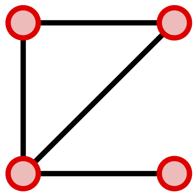
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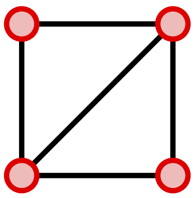
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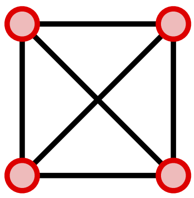

(3, 3, 3, 3)

[6 marks]

Answer 3

(i) Let the number of vertices and edges in a simple graph G be n and m respectively and label the vertices with the numbers 1, 2, ..., n.
Let the degree at each vertex be denoted by $d_1, d_2, ..., d_n$.
Consider each edge in turn.
Each has two ends that terminate at two distinct vertices.

Thus the total degree count $\sum_1^n d_i$ is increased by 2 by each edge.

As there are m edges altogether the desired result follows, that

$$\sum_1^n d_i = 2m \qquad \square$$

[4 marks]

(ii) A graph can be used to represent a gathering of people shaking hands. Each person at the gathering is represented by a vertex. An edge between two vertices indicates those two people have shaken hands. The degree of a vertex then denotes the total number of distinct people that the associated person has shaken hands with. Of course, one handshake (one edge) involves two people (two vertices) and so the total number of people who have shaken hands is twice the number of distinct handshakes that have taken place.

[2 marks]

(iii)

Lemma 1.1 : The Even Number of Odds
Every graph has an even number of odd-degree vertices.

Any graph can be considered to be constructed from p vertices of even degree and q vertices of odd degree. Label the vertices of the graph such that the even degree vertices are $d_1, d_2, ..., d_p$

and the odd degree vertices are $d_{p+1}, d_{p+2}, ..., d_{p+q}$

The total degree count is then given by, $\sum_1^p d_i + \sum_{p+1}^{p+q} d_i$

Regardless of whether p is odd or even, the sum of all the even degree vertices will be even. This is because both an odd number of even numbers and an even number of even numbers is even.

By way of deriving a contradiction, suppose that a graph has an odd number of vertices, q, of odd degree. Now, an odd number of odd numbers is odd. So we have a total degree count that is the sum of an even and an odd number which is odd. However, from the handshake lemma we know that the total degree count must be even.

This contradiction shows that the assumption that there can be an odd number of vertices, q, of odd degree is false. Thus we deduce that the number of vertices, q, of odd degree must be even. $\square$

[5 marks]

Answer 4

(i) From the handshaking lemma we can determine m the number of edges,

$$\sum_{1}^{20} d_i = 2m$$

$$20 \times 3 = 2m \qquad \text{As each of the 20 vertices is of degree 3}$$

$$m = 30 \text{ edges}$$

Now, from Euler's Polyhedral Formula, the number of faces is found,

$$n - m + f = 2$$

$$20 - 30 + f = 2$$

$$f = 12 \quad \text{faces}$$

[4 marks]

(ii) Generalising the part (i) answer,

$$np = 2m \iff m = \frac{np}{2}$$

$$n - \frac{np}{2} + f = 2 \iff f = 2 - n + \frac{np}{2}$$

This shows that either the regularity or the number of vertices must be even as $n, m, f, p \in \mathbb{Z}^+$ $\square$

[4 marks]

Answer 5

Lemma 1.2 : Edge Count of a Complete Graph (A Clique)

The complete graph K_n has exactly $\dfrac{n(n-1)}{2}$ edges

Proof (by induction)

Let $m(n) = \dfrac{n(n-1)}{2}, \quad n \in \mathbb{Z}^+$

where n is the number of vertices of K_n and $m(n)$ the corresponding number of edges.

When $n = 1, \quad m(1) = \dfrac{1(1-1)}{2}$

$= 0$ which is clearly true.

(The complete graph with one vertex, K_1, has no edges)

Assume that when $n = k, \ m(k) = \dfrac{k(k-1)}{2}$

If one additional vertex is added, it must connect to the k existing vertices.

In consequence,
$$\begin{aligned} m(k+1) &= m(k) + k \\ &= \frac{k(k-1)}{2} + k \\ &= \frac{k^2 - k}{2} + \frac{2k}{2} \\ &= \frac{k^2 + k}{2} \\ &= \frac{(k+1)k}{2} \\ &= \frac{(k+1)((k+1)-1)}{2} \end{aligned}$$

Which is precisely the assumed formula for $m(k)$ with k replaced with $k+1$

Therefore, if $m(n)$ is the number of edges when $n = k$,

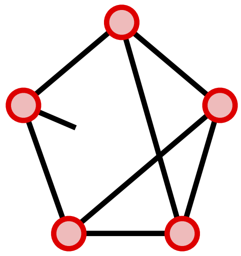
then $m(n)$ is also the number of edges when $n = k + 1$

As $m(n)$ is the number of edges when $n = 1$, $m(n)$ is also the number of edges for all $n \in \mathbb{Z}^+$ by mathematical induction. □

[6 marks]

Answer 6

(i) On trying to draw a 3-regular graph on 5 vertices, it's immediately apparent that it can't exist;



From the Handshaking Lemma (See question 3(i)) we know that the sum of all vertex degrees is twice the sum of the edges. We are trying to create graph that will have a vertex sum of 5×3 but this is twice 7.5 edges and so can't exist. In the figure you can literally see the 7.5 edges where the 0.5 edge has "nowhere to go".
Alternatively, invoke the result from question 3(iii) which stated that the number of vertices of odd degree must be even. We are trying to create a graph where the number of vertices of odd degree is odd.

[3 marks]

(ii) The above reasoning will apply to a r-regular graph on n vertices where the degree sum is given by rn and the number of edges by half of that. If r and n are both odd then their product is also odd and so not divisible by 2 which indicates that there will then be an edge with "nowhere to go".

That is, $m = \dfrac{rn}{2}$ is not an integer number of edges when r and n are odd.

[3 marks]

Answer 7

Lemma 1.3 : Vertex Pair Of Equal Degree
In any finite simple graph with at least two vertices, there must be at least two vertices which have the same degree.

Proof (*By contradiction*)
First a reminder that in a *simple* graph there is, at most, one edge between any vertex pair. In a graph with n vertices, a vertex can, at most, connect to all of the other $(n - 1)$ vertices. Suppose that there exists a graph with n vertices that have the different degrees. Counting down from the maximum degree of $(n - 1)$ these must be given by,

$$\{ d_1, d_2, d_3, ..., d_n \} = \{ 0, 1, 2, ..., n - 1 \}$$

However, this requires that the graph have a vertex with degree $(n - 1)$ and another of degree 0 which is a contradiction because the vertex of degree $(n - 1)$ has to connect to all other vertices. Thus there must be at least two vertices of the same degree. □

[4 marks]

Answer 8

It is the wording of this question that is most likely to cause difficulty !

"Let G be a graph with nine vertices such that each vertex is of either degree 5 or of degree 6. Show that G has at least six vertices of degree 5, or at least five vertices of degree 6".

By way of understanding the question let's initially take an unsophisticated approach and say that the nine vertices could divide between those of degree 5 and those of degree 6 as follows,

$$\begin{array}{c} 9 \times 5 + 0 \times 6 \\ 8 \times 5 + 1 \times 6 \\ 7 \times 5 + 2 \times 6 \\ 6 \times 5 + 3 \times 6 \\ 5 \times 5 + 4 \times 6 \\ 4 \times 5 + 5 \times 6 \\ 3 \times 5 + 6 \times 6 \\ 2 \times 5 + 7 \times 6 \\ 1 \times 5 + 8 \times 6 \\ 0 \times 5 + 9 \times 6 \end{array}$$

The top 4 rows of this table are the "at least six vertices of degree 5" and the bottom 5 rows are the "at least five vertices of degree 6".

So the only row that is not covered and for which a reason needs to be given for its removal is the 5 vertices of degree 5 and 4 vertices of degree 6.

This is readily done using the fact from question 3(iii) that the number of vertices in a graph of odd degree must be even.

After applying this criteria the table becomes,

$$\begin{array}{c} 8 \times 5 + 1 \times 6 \\ 6 \times 5 + 3 \times 6 \\ 4 \times 5 + 5 \times 6 \\ 2 \times 5 + 7 \times 5 \\ 0 \times 5 + 9 \times 6 \end{array}$$

and each row does now indeed satisfy the claim in the question. □

[4 marks]

Answer 9

In order to maximise the number of possible edges, we need to minimise the amount of disconnectedness. This will be achieved by having a graph that is only in two separated pieces, each of those pieces being maximally connected. Let k be the number of vertices in the first such piece and $(n - k)$ be the number of vertices in the second piece (such that the sum of all the vertices in the two pieces is n). From question 5 it is known that the complete graph K_n has exactly $\frac{n(n-1)}{2}$ edges. So the first piece has $\frac{k(k-1)}{2}$ edges and the second piece has $\frac{(n-k)(n-k-1)}{2}$ edges. In total the number of edges in given by,

$$\begin{aligned} E(k) &= \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\ &= \frac{k^2 - k + n^2 - nk - n - nk + k^2 + k}{2} \\ &= \frac{2k^2 - 2nk - n + n^2}{2} \end{aligned}$$

For any given graph, n is a fixed constant, and it is k that varies.

In fact, $1 \leqslant k \leqslant n - 1$

$E'(k) = 2k - n$ and this gives that $E(k)$ is a minimum when $k = \frac{n}{2}$

In other words, the least number of edges is obtained by having as close to half of the vertices in each piece of the disconnected half.

The maximum number of edges, which is what we are after, will thus occur at the extremes of the inequality for k (as it's a right way up quadratic, $\cup$).

When $k = 1$, $$\begin{aligned} E(1) &= \frac{2 - 2n - n + n^2}{2} \\ &= \frac{(n-2)(n-1)}{2} \end{aligned}$$

When $k = n - 1$, $$\begin{aligned} E(n-1) &= \frac{2(n-1)^2 - 2n(n-1) - n + n^2}{2} \\ &= \frac{2n^2 - 4n + 2 - 2n^2 + 2n - n + n^2}{2} \\ &= \frac{n^2 - 3n + 2}{2} \\ &= \frac{(n-2)(n-1)}{2} \end{aligned}$$

Thus, the maximum number of edges in a disconnected graph occurs when there is a single isolated vertex, the remaining vertices forming a complete graph with the number of edges given by $K_{n-1} = \frac{(n-2)(n-1)}{2}$ $\square$

[6 marks]

Answer 10

(i) This can be proven using induction.

Let $s(x)$ denote the number of subsets that a set S of x elements can have.

The result to be proven is that $s(x) = 2^x$

As the base case suppose we have a set of one element.
There are two possible subsets, either the empty set, or the set with the element in it.

Thus $s(1) = 2^1 = 2$ is established as a basis for the induction.

Assume that $s(k) = 2^k$ for some positive integer value of k.
Now consider enlarging the number of elements of set S by one.

For each of the subsets of $s(k)$ a pair of subset will be counted by

$s(k + 1)$ one from adding the new element and one from not adding it.

In consequence the count for $s(k + 1)$ will be twice that of $s(k)$.

$$\begin{aligned} s(k+1) &= 2 \times s(k) \\ &= 2 \times 2^k \qquad \text{by assumption} \\ &= 2^{k+1} \end{aligned}$$

which is precisely the assumed formula for $s(k)$ with k replaced with $k + 1$

Therefore, if $s(n)$ is the number of subsets when $n = k$,

then $s(n)$ is also the number of subsets when $n = k + 1$

As $s(n)$ is the number of subsets when $n = 1$, $s(n)$ is also the number of

subsets for all $n \in \mathbb{Z}^+$ by mathematical induction.

[3 marks]

(ii) Let M_{MAX} be the maximum possible number of edges, and label these.
Between any pair of vertices there is either an edge or there is not.
Determining the number of possible graphs now corresponds to determining the number of possible subsets for a set of x elements which, from part (i) is given by 2^x. That is $2^{M_{MAX}}$.
The graph on n vertices with the maximum possible number of edges is the complete graph K_n. From question 5,

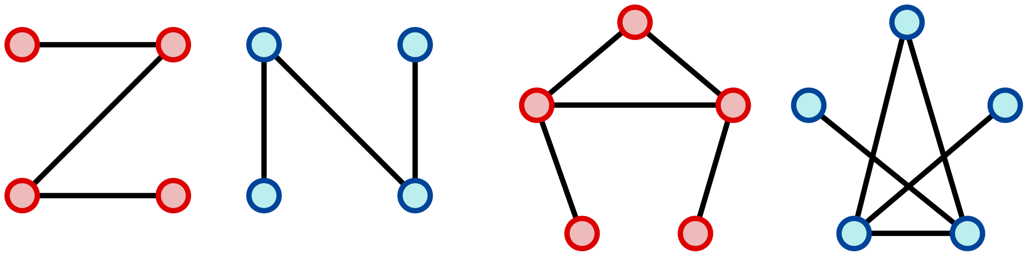
"the complete graph K_n has exactly $\dfrac{n(n-1)}{2}$ edges"

Thus there are exactly $2^{\frac{1}{2}n(n-1)}$ labelled simple graphs on n vertices. □

[3 marks]

Answer 11

(i)



(1, 1, 2, 2) (1, 1, 2, 2) (1, 1, 2, 3, 3) (1, 1, 2, 3, 3)

[2 marks]

(ii) On a graph of n vertices, the maximum degree at any vertex is $n - 1$ which occurs when that vertex has an edge going to each of the other $n - 1$ vertices. In section 1.6 it was observed that "the graph sum $G + \overline{G}$ on graph of degree n is the complete graph K_n".

So, if the degree of vertex i is d_i and the degree of its complement $\overline{d_i}$

then $d_i + \overline{d_i} = n - 1$

Also, it must be the case that,

$$\begin{aligned} \overline{d_1} &= d_n, \\ \overline{d_2} &= d_{n-1} \\ \dots &= \dots \\ \overline{d_i} &= d_{n-(i-1)} \end{aligned}$$

That is, $\overline{d_i} = d_{n+1-i}$

Combining the two results gives that $d_i + d_{n+1-i} = n - 1$

[2 marks]

(iii) From question 5,

$$\text{“}K_n \text{ has exactly } \frac{n(n-1)}{2} \text{ edges”}$$

is obtained the fact that the complete graph K_5 will have 10 edges.
It is then deduced that a self-complementary graph on n vertices will have half of this, which for $n = 5$ is 5 edges.
The handshaking lemma then states that the sum of degrees will be double this, which in this case is 10.
The part (ii) symmetry result implies that the degree sequence must be,

$$(d_1, d_2, d_3, 4 - d_1, 4 - d_2)$$

In combination we have that,

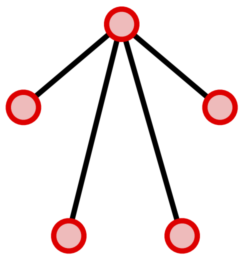
$$d_1 + d_2 + d_3 + 4 - d_1 + 4 - d_2 = 10$$

$$\text{from which we get that } d_3 = 2$$

The possibilities are now,

case	d_1	+	d_2	+	d_3	+	d_4	+	d_5
1	0	+	0	+	2	+	4	+	4
2	0	+	1	+	2	+	3	+	4
3	0	+	2	+	2	+	2	+	4
4	1	+	1	+	2	+	3	+	3
5	1	+	2	+	2	+	2	+	3
6	2	+	2	+	2	+	2	+	2

This list can be pruned further,
Cases 1, 2 and 3 require that we have a graph on 5 vertices that has a vertex of degree 4 and one of degree 0 which is clearly not possible as illustrated by the diagram below.



Out of interest, readers may like to note that Case 4 is satisfied by the graph on 5 vertices from part (i) which is the only graph (up to isomorphism) to do so. Case 6 is satisfied by C_5 as noted is section 1.6 which, again, is the only graph (up to isomorphism) to do so.
Case 5 yields no self-complementary graphs.

[4 marks]

(iv) The idea here is to generalising slightly the approach taken in part (iii) starting again will the result from question 5 that

$$m(K_n) = \frac{n(n-1)}{2}$$

Again it is deduced that a self-complementary graph on n vertices, G_n, will have half that number of edges,

$$m(G_n) = \frac{n(n-1)}{4}$$

As n and $(n-1)$ represent two consecutive integers they cannot both be even. Furthermore, $m(G_n)$ must be an integer and so either n is divisible by 4 or $(n-1)$ is divisible by 4. In other words the number of vertices of a self-complementary graph, $n(G_n)$, (or just n) has the form $4k$ or $4k+1$ for $k \in \mathbb{Z}^+$ which is the result requested. ☐

[5 marks]

(v) Case 1 : For a graph of order $4k$, from part (iv),

$$\frac{4k(4k-1)}{4} = k(4k-1) \text{ edges}$$

From the handshaking lemma, the sum of the degrees of all vertices will be $2k(4k-1)$ but this cannot be shared equally among $4k$ vertices. That is,

$$\frac{2k(4k-1)}{4k} = \frac{4k-1}{2} \text{ with } k \in \mathbb{Z} \text{ is not an integer.}$$

Thus, $n \neq 0 \pmod 4$

Case 2 : For a graph of order $4k+1$, from part (iv),

$$\frac{(4k+1)(4k+1-1)}{4} = k(4k+1) \text{ edges}$$

From the handshaking lemma, the sum of the degrees of all vertices will be $2k(4k+1)$ which can be shared equally among the $4k+1$ vertices. Thus, $n \equiv 1 \pmod 4$

Overall we conclude that if G is a self-complementary regular graph on n vertices then $n \equiv 1 \pmod 4$ ☐

[5 marks]