Lesson 8

Further A-Level Pure Mathematics, Core 2
Differential Equations II

8.2 Consolidation #2

Abbreviated summary;

- Separating the variables can solve some first-order differential equations.

- A first-order differential equation of the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$ can be solved by multiplying every term by the integrating factor $I = e^{\int P(x)\,dx}$

- The nature of the roots α and β of the auxiliary equation determine the general solution to the second order differential equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$
  The general solution depends on the auxiliary equation's discriminant, D;
  ◇ Case 1, $\boldsymbol{D > 0}$: $y = Ae^{\alpha x} + Be^{\beta x}$, for arbitrary A, B.
  ◇ Case 2, $\boldsymbol{D = 0}$: $y = (A + Bx)\,e^{\alpha x}$, for arbitrary A, B.
  ◇ Case 3, $\boldsymbol{D < 0}$: $y = e^{px}\,(A\cos qx + B\sin qx)$, for arbitrary A, B.

- Particular Integral suggestions,

Form of $f(x)$	Form of PI
$u\,x^r + \ldots + vx + w$	$U\,x^r + \ldots + Vx + W$
$u\cos kx + v\sin kx$	$U\cos kx + V\sin kx$
$u\,e^{kx}$	$U\,e^{kx}$

- But watch out for...

Clash of Function : An "Advice" Algorithm

When solving an equation of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),\ \text{where } a, b \text{ and } c \text{ are constants,}$$

and having obtained the complementary function,
if a piece, $g(x)$, of a proposed particular integral, $p(x)$, has already occurred in the complementary function, modify the particular integral by multiplying it by x. That is, replace $p(x)$ with $x\,p(x)$.
Repeat this process, if necessary, until there is no piece in common in any part of the complementary function or particular integral.

8.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Find the general solution to the differential equation,

$$y'' + 2\sqrt{2}\,y' + 2y = 10$$

[5 marks]

Question 2

Find the general solution to the differential equation,

$$y'' - y' - 12y = 144x$$

[7 marks]

Question 3

Find the general solution to the differential equation,

$$y'' - 7y' + 10y = x\,e^{x}$$

Assume the particular integral is of the form $(Ux + V)\,e^{x}$

[10 marks]

Question 4

(i) Find the value of U for which $y = Ue^{2x}$ is a particular integral of the differential equation $y'' - 4y' + 13y = e^{2x}$

[4 marks]

(ii) Using your answer to part (i), find the general solution to the differential equation.

[5 marks]

Question 5

The differential equation $y'' - 4y' + 4y = 4e^{2x}$ is to be solved.

(i) Find the complementary function.

[3 marks]

(ii) Explain why neither Ue^{2x} nor $Ux\,e^{2x}$ can be a particular integral.

[2 marks]

A particular integral has the form $Ux^2\,e^{2x}$

(iii) Determine the value of the constant U and find the general solution.

[6 marks]

Question 6

Cameron, who is skilled in the dark arts, is about to advise you on the particular integral to use when solving the differential equation,

$$y'' + y = 3\,sin\,2x$$

Although your natural inclination is to assume a particular integral of the form $y = U sin\,2x + V cos\,2x$, Cameron assures you that $U sin\,2x$ is sufficient.

(i) Given that Cameron speaks wise words, determine the value of *U*.

[4 marks]

(ii) Using your answer to part (i), find the general solution to the differential equation.

[4 marks]