Lesson 5

Further A-Level Pure Mathematics, Core 2

Differential Equations II

5.1 Consolidation #1

Here is a summary of the previous four lessons. There is more to come but it's important to consolidate the techniques covered so far.

- Separating the variables can solve some first-order differential equations.

- A first-order differential equation of the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ can be solved by multiplying every term by the integrating factor $I = e^{\int P(x)\,dx}$

- The nature of the roots α and β of the auxiliary equation determine the general solution to the second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

  The general solution depends on the auxiliary equation's discriminant, D;

  ◇ Case 1, $D > 0$

  Two distinct real roots: $y = Ae^{\alpha x} + Be^{\beta x}$, for arbitrary constants, A, B.

  ◇ Case 2, $D = 0$

  One repeated root: $y = (A + Bx)\,e^{\alpha x}$, for arbitrary constants, A, B.

  ◇ Case 3, $D < 0$

  Two complex conjugate roots $\alpha = p + q\mathrm{i}$, $\beta = p - q\mathrm{i}$

  $y = e^{px}(A\cos qx + B\sin qx)$, for arbitrary constants, A, B.

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 38

Question 1

Given that $y = 1$ when $x = 0$, find the particular solution to the differential equation,

$$\frac{dy}{dx} = y\sinh x$$

[4 marks]

Question 2

(i) Let $M = \int e^{-3x} \sin x \; dx$

By using integration by parts twice show that,

$10M = -e^{-3x}(\cos x + 3\sin x) + c$, for some constant c

[4 marks]

(ii) $\dfrac{dy}{dx} - 3y = \sin x$

Given that $y = 0$ when $x = 0$, find y in terms of x.

[4 marks]

Question 3

Find the solution to the differential equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

given that, when $x = 0$, both $y = 0$ and $\frac{dy}{dx} = 3$

[8 marks]

Question 4

Find y in terms of k and x, given that

$$\frac{d^2y}{dx^2} + k^2 y = 0, \text{ where } k \text{ is a constant}$$

and $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$

[8 marks]

Question 5

(i) Find the general solution to the differential equation,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

[4 marks]

(ii) Given that $x = 1$ and $\frac{dx}{dt} = 1$ at $t = 0$, find the particular solution to the differential equation, giving your answer in the form $x = f(t)$

[2 marks]

(iii) Write your part (ii) answer in the form $R\,e^{kt}\,cos\,(2t\ -\ \alpha)$ where k, R and α are constants that you have determined the exact value of.

[2 marks]

(iv) Sketch the curve with equation $x = f\,(t)\,,\ \ 0\ \leqslant\ t\ \leqslant\ \pi,$ showing the coordinates, as multiples of π, of the points where the curve cuts the t-axis.

[2 marks]