3.1 Integrating Factor Modulus Questions

When the method of solving a differential equation involves an integrating factor, the modulus function is frequently involved. Quite often (even in textbooks) there is a fudge along the following lines,

$$\int P(x)\ dx = 3\int \frac{1}{x}\ dx = 3\,ln\,(x)$$

where the modulus signs are mysteriously lost along with the constant of integration. Unfortunately the sloppiness is rewarded; it gets the correct answer ! In these lessons, however, we will include the proper logical steps which hinge upon realising (for example) that $|y| = Bx^2$ where B is a real positive number implies that $y = Ax^2$ where A is a real number.

3.2 Example #1

To solve $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{e^x}{x^3}$ first identify that $P(x) = \dfrac{3}{x}$ and $Q(x) = \dfrac{e^x}{x^3}$

$$\int P(x)\ dx = 3\int \frac{1}{x}\ dx = 3\,ln|x| + c = ln|x^3| + c$$

$$\begin{aligned} I &= e^{\int P(x)\,dx} \\ &= e^{ln|x^3| + c} \\ &= |x^3|\ e^c \\ &= B\,|x^3| \quad \text{for } B \in \mathbb{R},\ B > 0 \\ &= A\,x^3 \qquad \text{for } A \in \mathbb{R} \end{aligned}$$

As both sides of the differential equation are about to be multiplied by I there is no need for the constant of integration, the "A". If it were included then it would be immediately cancelled from both sides.

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

$$x^3\frac{dy}{dx} + 3x^2 y = e^x$$

integrate both sides with respect to x

$$\int x^3\frac{dy}{dx} + 3x^2 y\ dx = \int e^x\ dx$$

$$x^3 y = e^x + c$$

$$y = \frac{e^x + c}{x^3} \quad \text{is the general solution}$$

[6 marks]

3.3 Example #2

To solve $\dfrac{dy}{dx} + y\,cot\,(x) = csc\,(x) \quad x,\ y \in \mathbb{R},\ x \neq 180n,\ n \in \mathbb{Z}$

note that $P\,(x) = cot\,(x)$ and $Q\,(x) = csc\,(x)$

$$\int P\,(x)\ dx = \int cot\,(x)\ dx = ln \left|\ sin\,(x)\ \right| + c$$

$$\begin{aligned} I &= e^{\int P(x)\,dx} \\ &= e^{ln|\,sin(x)\,| + c} \\ &= B\left|\ sin\,(x)\ \right| \quad \text{for } B \in \mathbb{R},\ B > 0 \\ &= A\,sin\,(x) \qquad \text{for } A \in \mathbb{R} \end{aligned}$$

$$\frac{dy}{dx} + y\,cot\,(x) = csc\,(x)$$

Multiplying through by I and cancelling the A gives,

$$sin\,(x)\,\frac{dy}{dx} + y\,\frac{cos\,(x)}{sin\,(x)} \times sin\,(x) = \frac{1}{sin\,(x)} \times sin\,(x)$$

$$sin\,(x)\ \frac{dy}{dx} + cos\,(x)\ y = 1$$

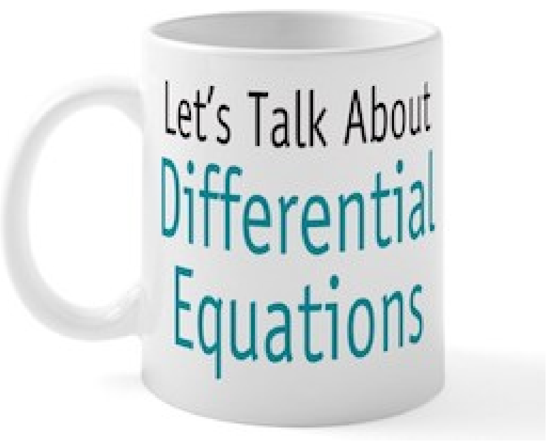
integrate both sides with respect to x

$$\int sin\,(x)\ \frac{dy}{dx} + cos\,(x)\ y\ dx = \int 1\ dx$$

$$sin\,(x)\ y = x + c \qquad \text{for any real constant } c$$

$$y = \frac{x + c}{sin\,(x)} \qquad \text{is the general solution}$$

[6 marks]



3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

(i) By means of an integrating factor, find the general solution of,

$$\frac{dy}{dx} + \frac{y}{(x+1)} = 1, \qquad x, y \in \mathbb{R}, \; x \neq -1$$

Give an exact answer in the form $y = f(x)$

[6 marks]

(ii) Find the particular solution for which $y = 1$ when $x = 1$

[2 marks]

Question 2

(i) By means of an integrating factor, find the general solution of,

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad x, y \in \mathbb{R},\ x \neq 0$$

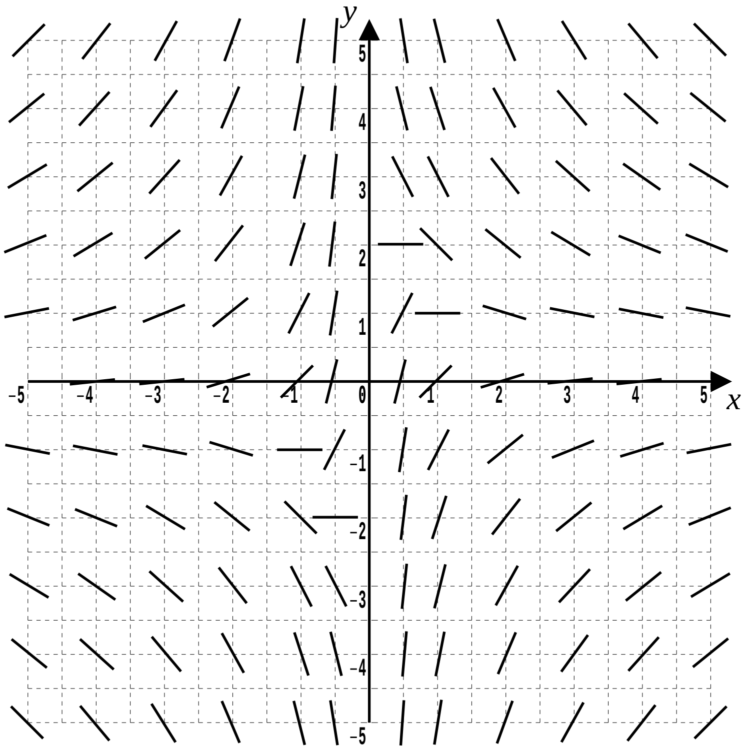
Give an exact answer in the form $y = f(x)$

[6 marks]

(ii) Find the particular solution for which $y = 1$ when $x = 2$

[2 marks]

(iii) Sketch your part (ii) solution on this slope field plot.



[2 marks]

Question 3

A-Level Examination Question from June 2015, Paper FP2, Q3 (Edexcel)

Find, in the form $y = f(x)$, the general solution of the differential equation,

$$tan(x)\ \frac{dy}{dx} + y = 3\,cos(2x)\ tan(x), \qquad 0 < x < \frac{\pi}{2}$$

[6 marks]

Question 4

A-Level Examination Question from June 2018, Paper F2, Q2 (Edexcel)

(a) Find the general solution of the differential equation,

$$\left(x^2 + 1\right)\frac{dy}{dx} + xy - x = 0$$

giving your answer in the form $y = f(x)$

[6 marks]

(b) Find the particular solution for which $y = 2$ when $x = 3$

[2 marks]