Undergraduate Lectures in Mathematics
A First Year Course

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MATRIX ALGEBRA

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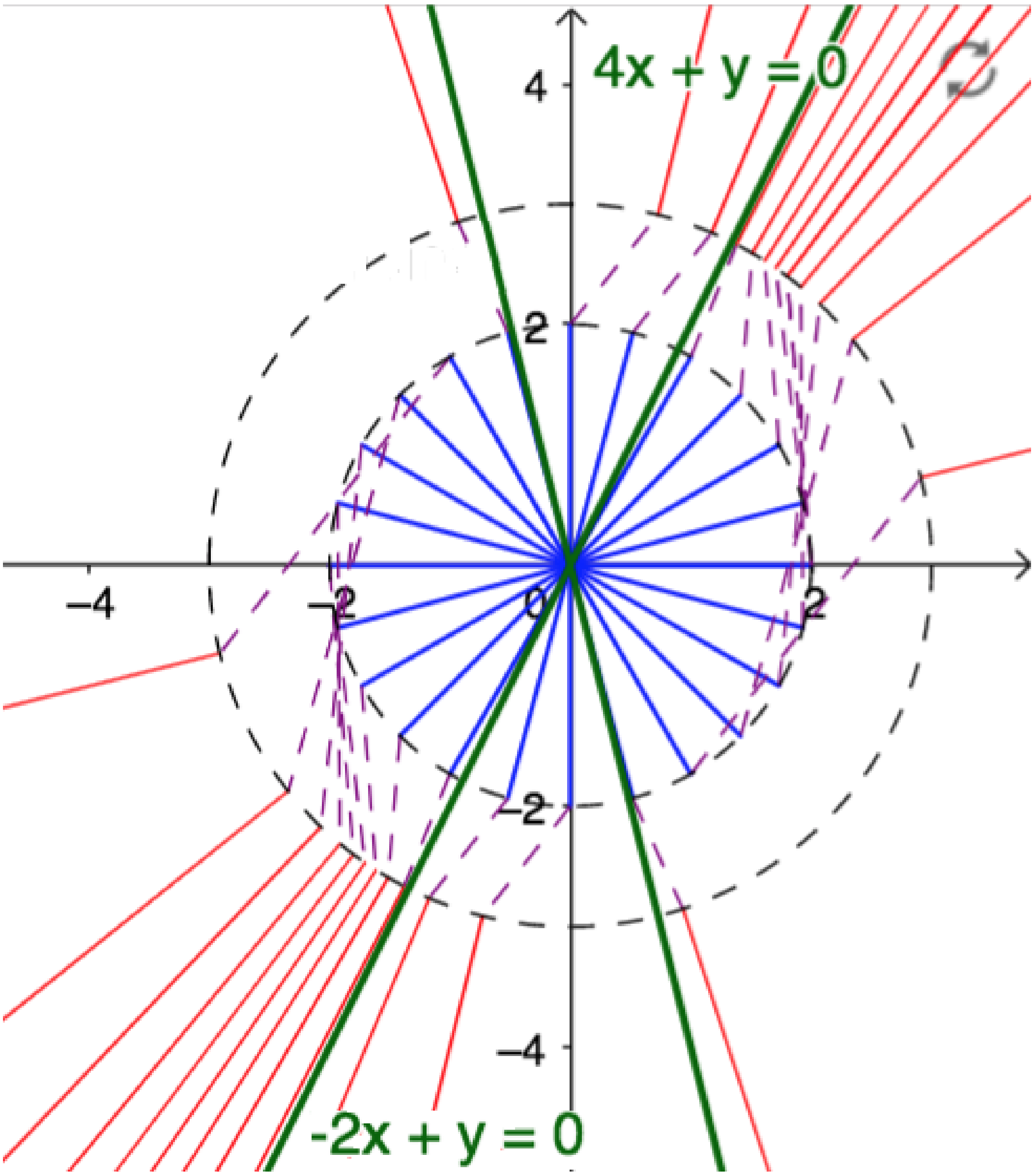


GeoGebra's "Eigenpicture" for the matrix $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$ showing how lines through the origin are given varying amounts of rotation about (0, 0) by the matrix.

Two special invariant lines, not rotated by the matrix, are highlighted in green.

MATRIX ALGEBRA

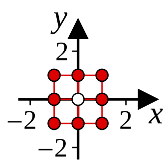
Lecture 1

Undergraduate Lectures in Mathematics : First Year

Matrix Algebra

1.1 Surface Distortion : First Experiment

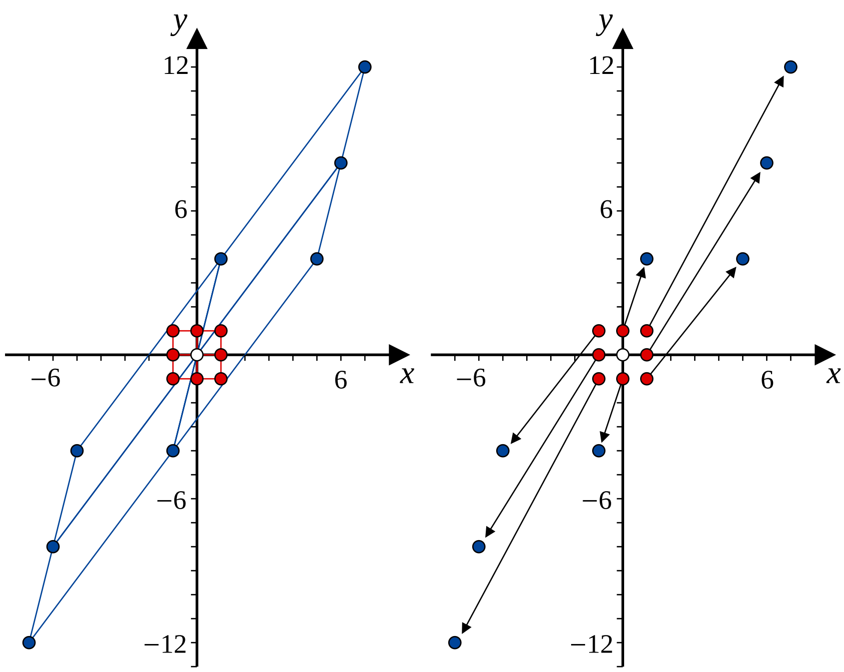
When a 2×2 matrix is applied to a point (other than the origin) it moves the point. A surface is made up of points and so it is natural to wonder about the bigger picture, namely, what is the matrix doing to the surface. To get an impression of how a matrix is distorted the surface it could be applied to a grid of four squares with corners specified by nine points.



As an example, let's apply the matrix $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$ to this grid of nine points.

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

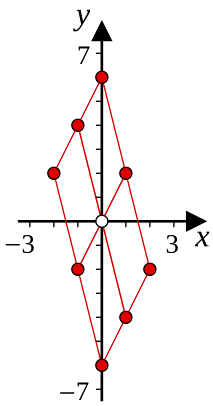
$$= \begin{pmatrix} -5 & 1 & 7 & -6 & 0 & 6 & -7 & -1 & 5 \\ -4 & 4 & 12 & -8 & 0 & 8 & -12 & -4 & 4 \end{pmatrix}$$



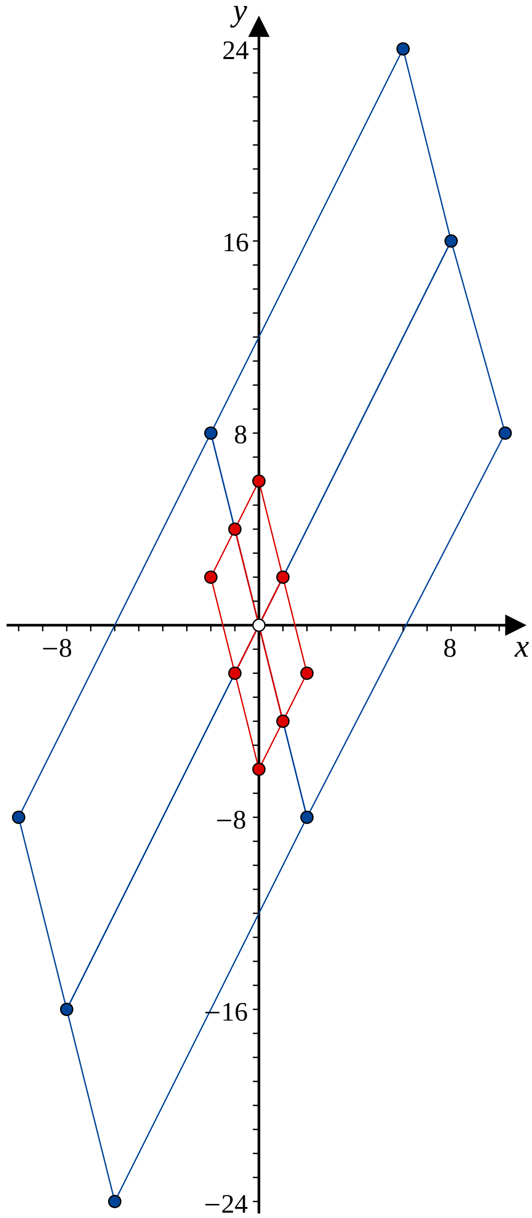
The left diagram gives an impression of how the grid has been distorted by our matrix **A**, whilst the right diagram shows how the individual points have moved. The diagrams suggest that **A** is expansive and that there is some sort of order to how the points are being moved. Clearly, something is going on but exactly what is not pinned down well by this first experiment.

1.2 Surface Distortion : Second Experiment

Progress is made by exploring other starting grids. Of those other grids the following is by far the most interesting.



This time, **A** has the following interesting effect,



This time there is an alignment between the red and the blue grids; The blue seems to be a bigger version of the red, although it is stretched more in one direction than the other. Here are the details of the calculations for this second experiment,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 2 \\ 2 & 4 & 6 & -2 & 0 & 2 & -6 & -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -2 & 6 & -8 & 0 & 8 & -6 & 2 & 10 \\ -8 & 8 & 24 & -16 & 0 & 16 & -24 & -8 & 8 \end{pmatrix}$$

1.3 Vectors and Scalings from The Experiments

In the first experiment the red grid was made up of copies of the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In the second experiment the red grid vectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ which can be normalized (made of unit length) as $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Typically, the red and the blue grids do not align and so the situation where they do makes the *eigenvectors* $\boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ special (where k_1 and k_2 are scaling factors that change the vector's length but not its direction).

Focussing further on the second experiment, in the $\boldsymbol{v}_1$ direction the red point (1, 2) moved to the blue point (8, 16) which is a scaling factor of 8. This is length scaling factor is known as an *eigenvalue* λ_1. In the $\boldsymbol{v}_2$ direction the red point (1, – 4) moved to the blue point (2, – 8) which is a length scaling factor of 2. This gives another *eigenvalue* λ_2. Eigen is a German word for "special". Having seen geometrically why the eigenvectors and the eigenvalues of a matrix are very special, attention can turn to how to find them algebraically.

1.4 Eigenvalues and Eigenvectors of a Square Matrix

Definition : Eigenvalues and Eigenvectors

An eigenvector of a matrix **A** is a non-zero column vector $\boldsymbol{x}$ which satisfies the equation $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ where λ is a scalar. The value of the scalar λ is the eigenvalue of the matrix **A** corresponding to the eigenvector $\boldsymbol{x}$.

So, for the matrix $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$ we can write that,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

which confirms that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector associated with eigenvalue 8, and that $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is an eigenvector associated with eigenvalue 2.

More generally, from the definition,

$$\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$\mathbf{A}x = \lambda\mathbf{I}\boldsymbol{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = 0$$

As $\boldsymbol{x}$ is non-zero (by definition) the matrix $(\mathbf{A} - \lambda\mathbf{I})$ must be zero. In other words it must be singular. To be singular it must have a determinant of zero. Thus,

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This is key to finding the eigenvalues and eigenvectors of a matrix by algebra.

1.4.1 Example

Question: Use matrix algebra to find the eigenvalues and eigenvectors of $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$.

Solution:

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\left(\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$det\begin{pmatrix} 6-\lambda & 1 \\ 8 & 4-\lambda \end{pmatrix} = 0$$

$$(6-\lambda)(4-\lambda) - 8 \times 1 = 0$$

$$24 - 6\lambda - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0 \qquad \text{The characteristic equation}$$

$$(\lambda - 8)(\lambda - 2) = 0$$

$$\lambda_1 = 8 \text{ or } \lambda_2 = 2 \text{ are the eigenvalues}$$

For λ_1,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 6x + y \\ 8x + 4y \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \end{pmatrix}$$

Matching upper elements, $6x + y = 8x \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For λ_2,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 6x + y \\ 8x + 4y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

Matching upper elements, $6x + y = 2x \Rightarrow y = -4x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

where k_1 and k_2 are any constants.

1.5 The Cayley-Hamilton Theorem

The characteristic equation of a matrix is so called because, as we have seen, it captures some important features of the matrix. However, different matrices can have these same features in common, and so have the same characteristic equation. There is a remarkable theorem associated with the characteristic equation that is now stated without proof;

Theorem 1.1 : The Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

The theorem is now illustrated using this lecture's ongoing example, $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$.

This has the characteristic equation $\lambda^2 - 10\lambda + 16 = 0$.

The Cayley-Hamilton theorem therefore claims that $\mathbf{A}^2 - 10\mathbf{A} + 16\mathbf{I} = \mathbf{0}$.

$$\begin{aligned}
\text{LHS} &= \mathbf{A}^2 - 10\mathbf{A} + 16\mathbf{I} \\
&= \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}^2 - 10\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} + 16\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 44 & 10 \\ 80 & 24 \end{pmatrix} - \begin{pmatrix} 60 & 10 \\ 80 & 40 \end{pmatrix} + \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \mathbf{0} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

1.5.1 Example

Question: Use the Cayley-Hamilton theorem to find $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}^{-1}$

Answer: Previously seen;

$$\mathbf{A}^2 - 10\mathbf{A} + 16\mathbf{I} = \mathbf{0}$$

$$\mathbf{A} - 10\mathbf{I} + 16\mathbf{A}^{-1} = \mathbf{0} \qquad \text{by post-multiplying through by } \mathbf{A}^{-1}$$

$$\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{16}(10\mathbf{I} - \mathbf{A}) \\
&= \frac{1}{16}\left(10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\right) \\
&= \frac{1}{16}\begin{pmatrix} 4 & -1 \\ -8 & 6 \end{pmatrix}
\end{aligned}$$

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ maps the point (5, 2) to the point (− 15, − 6).

(i) Write down an eigenvalue of the matrix representing T.

[1 mark]

(ii) Write down a corresponding eigenvector.

[2 marks]

(iii) Write down the equation of the corresponding invariant line.

[1 mark]

Question 2

Show that any 2×2 matrix of the form $\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbb{R}$ has complex eigenvalues given by $a \pm b\mathbf{i}$

[4 marks]

Question 3

The matrix $\mathbf{A} = \begin{pmatrix} 3 & k \\ 1 & -1 \end{pmatrix}$ has a repeated eigenvalue.

(i) Find the value of k.

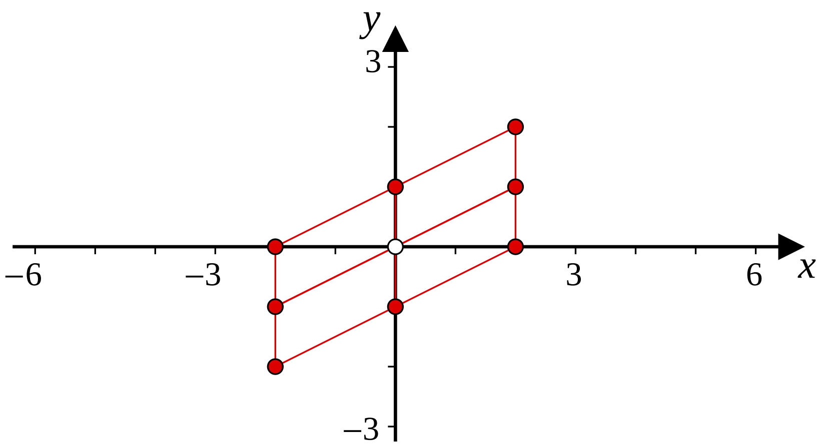
[4 marks]

(ii) Find the value of the repeated eigenvalue, an equation of a corresponding invariant line and a corresponding eigenvector.

[3 marks]

(iii) The red grid below is specified by $\begin{pmatrix} -2 & 0 & 2 & -2 & 0 & 2 & -2 & 0 & 2 \\ 0 & 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 \end{pmatrix}$.

Draw the blue grid that results then transforming this by **A**.



[4 marks]

(iv) Make two comments on your part (iii) answer.

[2 marks]

Question 4

A-Level Examination Question from May 2019, Paper FP2, Q1 (Edexcel)

Given that,

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

(a) find the characteristic equation for the matrix **A**.
Simplify your answer.

[2 marks]

(b) Hence find an expression for the matrix $\mathbf{A}^{-1}$ in the form $\lambda\mathbf{A} + \mu\mathbf{I}$, where λ and μ are constants to be found.

[3 marks]

Question 5

Let λ be an eigenvalue of a matrix **A**

Prove by induction that λ^n is an eigenvalue of $\mathbf{A}^n$, for all $n \in \mathbb{Z}^+$

[6 marks]

Question 6

A-Level Examination Question from June 2020, Paper FP2, Q3 (Edexcel)

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Show that, in terms of k, a characteristic equation for **M** is given by,

$$\lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$$

[3 marks]

Given that *det* **M** = 5

(b) **(i)** find the value of k

(ii) use the Cayley-Hamilton theorem to find the inverse of **M**

[7 marks]

Question 7

Show that any 2×2 matrix of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies its own characteristic equation and hence prove the Cayley-Hamilton theorem for 2×2 matrices.

[8 marks]