Lesson 3

3.1 A New Transformation

A transformation not encountered at GCSE level is that of inversion.

Given a function, $f\,(x)$, the inversion of this function is $\dfrac{1}{f\,(x)}$.

Although not recommended, this could be written as $[\, f\,(x)\,]^{-1}$ but notice that, in general, this is not the same as $f^{-1}\,(x)$ which is the inverse function.

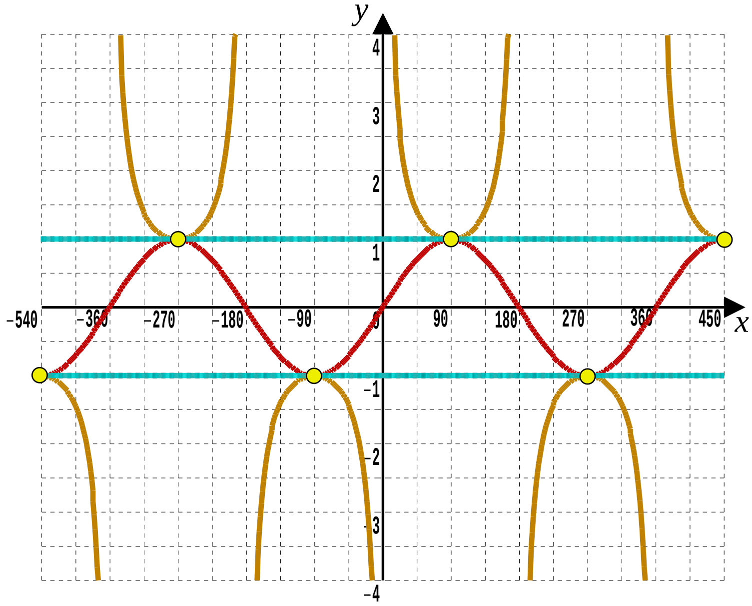
Inversion

The transformation inversion is the taking of the reciprocal of a given function.

Under inversion, $f\,(x) \;\rightarrow\; \dfrac{1}{f\,(x)}$

3.2 Example (Sine Function)

On the graph below, in red, is the familiar sine function, $f\,(x) \;=\; \sin x$ and, in gold, the inversion of this function $\dfrac{1}{f\,(x)} \;=\; \dfrac{1}{\sin x}$



Notice that,

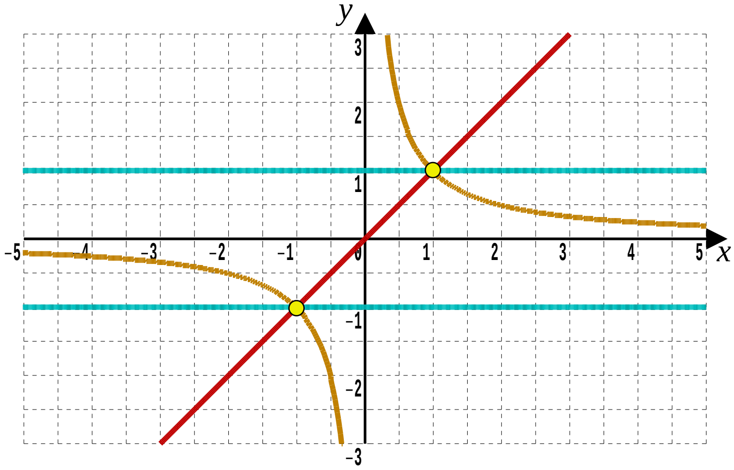
- as the reciprocal of ±1 is also ±1, points on the lines $y = \pm\, 1$ are invariant.
- the inversion has a vertical asymptote each time the sine function is zero.

☞ Draw in the vertical asymptotes on the graph.

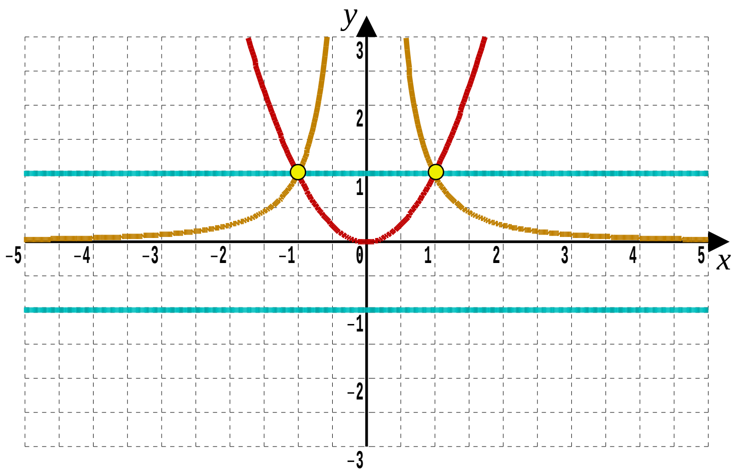
[1 mark]

3.3 Example (Power Function)

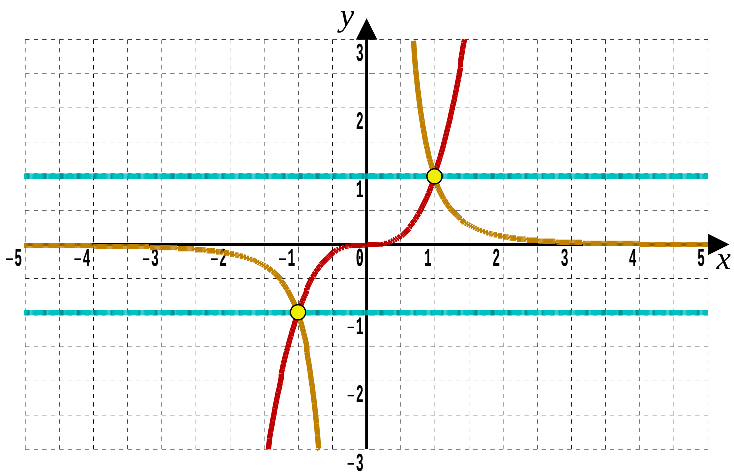
The power function, $f(x) = x^n$ has inversion, $\dfrac{1}{f(x)} = \dfrac{1}{x^n}$



$$n = 1 \;:\; y = x \;:\; y = \frac{1}{x}$$



$$n = 2 \;:\; y = x^2 \;:\; y = \frac{1}{x^2}$$



$$n = 3 \;:\; y = x^3 \;:\; y = \frac{1}{x^3}$$

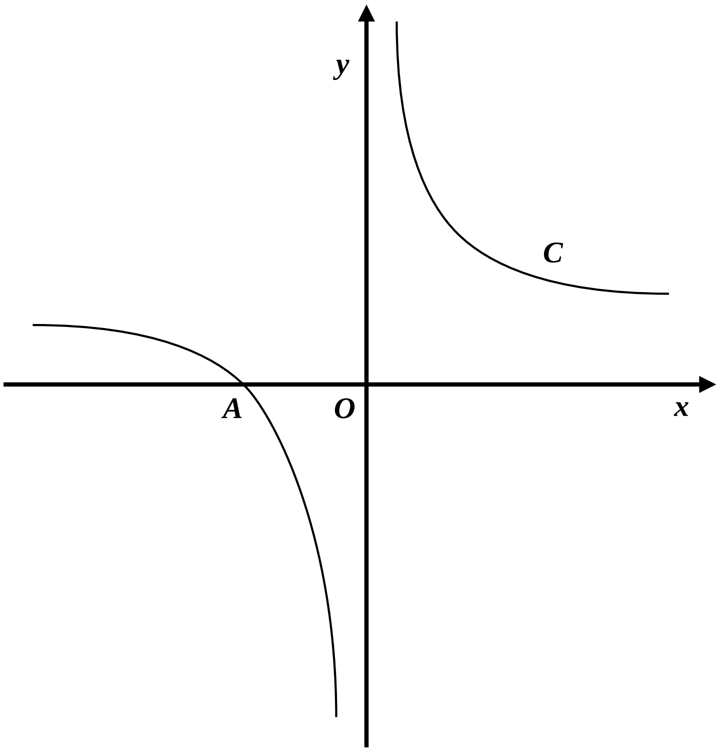
3.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 72

Question 1

A-Level C1 Examination Question from May 2014, Q4



The diagram shows a sketch of the curve C with equation

$$y = \frac{1}{x} + 1 \qquad x \neq 0$$

The curve C crosses the x-axis at the point A.

(a) State the x coordinate of the point A.

[1 mark]

The curve D has equation $y = x^2(x - 2)$ for all real values of x.

(b) Add a sketch of curve D to the diagram, above.
Show the coordinates of each point where the curve D crosses the axes.

[3 marks]

(c) Using your sketch, state, giving a reason, the number of real solutions to the equation

$$x^2(x - 2) = \frac{1}{x} + 1$$

[1 mark]

Question 2

A-Level Examination Question from January 2009, Paper C1, Q8 (Edexcel)

The point $P\,(1, a)$ lies on the curve with equation

$$y = (x+1)^2(2-x)$$

(a) Find the value of a

[1 mark]

(b) On the axes below sketch the curves with the following equations;

(i) $y = (x+1)^2(2-x)$ **(ii)** $y = \dfrac{2}{x}$

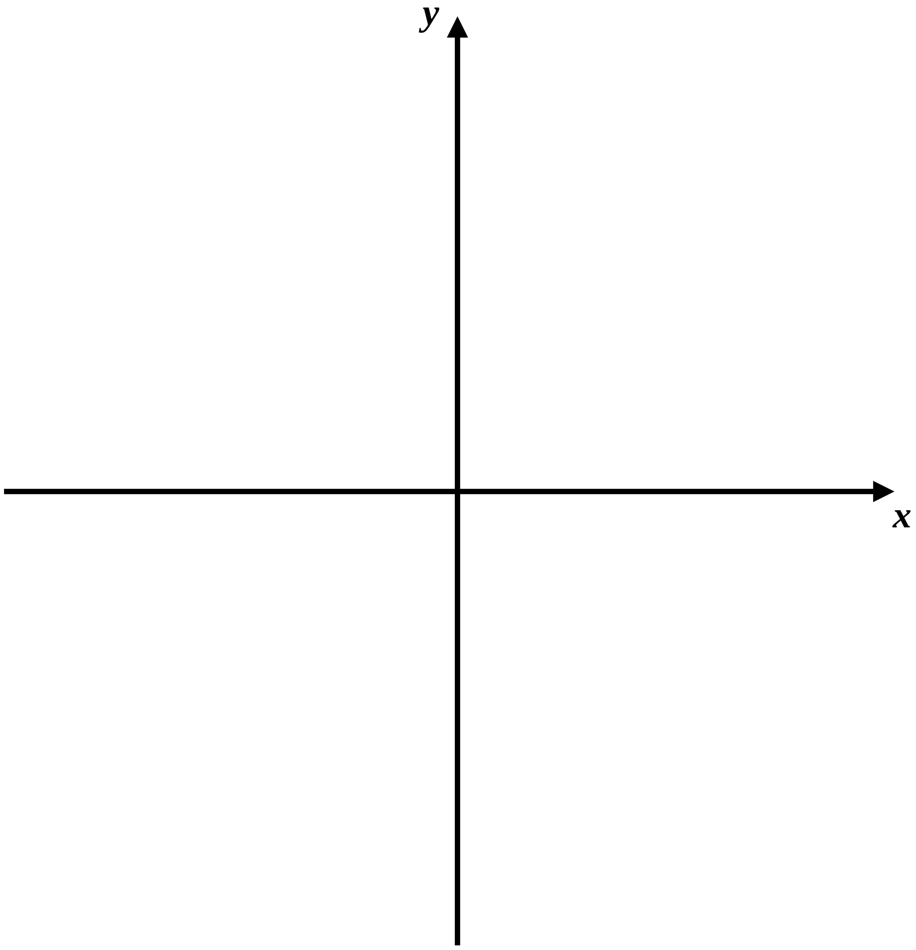
On your diagram show clearly the coordinates of any points at which the curves meet the axes.

[5 marks]

(c) With reference to your diagram in part (b) state the number of real solutions to the equation

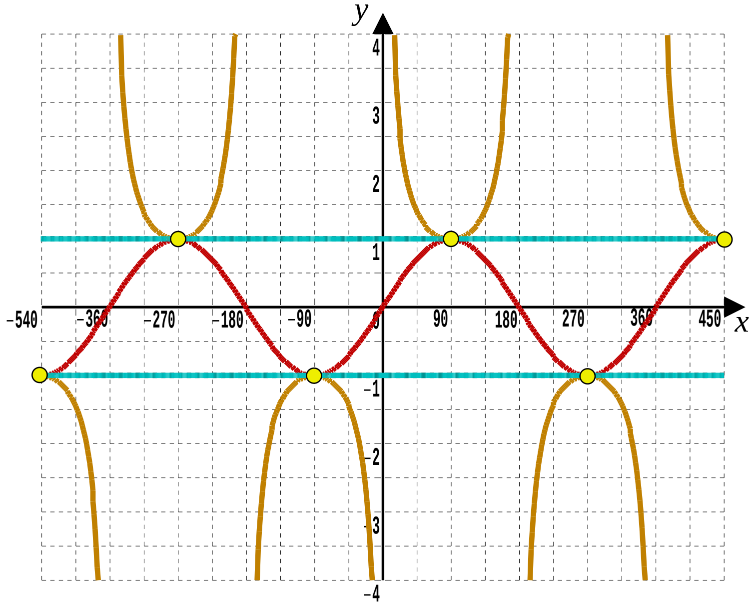
$$(x+1)^2(2-x) = \frac{2}{x}$$

[1 mark]



Question 3

Graphed is the inversion of the sine function, $f(x) = \sin x \rightarrow \dfrac{1}{f(x)} = \dfrac{1}{\sin x}$



Produce similar graphs for the inversion for,

(i) the cosine function, $f(x) = \cos x \rightarrow \dfrac{1}{f(x)} = \dfrac{1}{\cos x}$

(ii) the tangent function, $f(x) = \tan x \rightarrow \dfrac{1}{f(x)} = \dfrac{1}{\tan x}$

These inversions have names, $\dfrac{1}{\sin x} = \csc x$, $\dfrac{1}{\cos x} = \sec x$, $\dfrac{1}{\tan x} = \cot x$

[**3, 3 marks**]

Question 4

For the curve with equation

$$y = (x - 1)^2 - 16$$

(i) write down the coordinates of the minimum point,

[1 mark]

(ii) expand the brackets, and hence write down the coordinates of where the curve crosses the *y*-axis,

[1 mark]

(iii) factorise your part (ii) answer, and hence write down the coordinates of where the curve crosses the *x*-axis.

[1 mark]

(iv) Sketch the curve.

[2 marks]

Question 5

For the curve with equation

$$y = (x - 4)^2 + 1$$

explain, using mathematics, why it does not cross the *x*-axis.

[2 marks]

Question 6

A-Level Examination Question from January 2011, Paper C1, Q10 (Edexcel)

(a) Sketch the graphs of

(i) $y = x(x+2)(3-x)$ **(ii)** $y = -\dfrac{2}{x}$

Show clearly the coordinates of all points where the curves cross the coordinate axes.

[6 marks]

(b) Using your sketch state, giving a reason, the number of real solutions to the equation

$$x(x+2)(3-x) = -\frac{2}{x}$$

[2 marks]

Question 7

A-level Examination Question from January 2006, Paper C1, Q10 (Edexcel)

$$x^2 + 2x + 3 \equiv (x + a)^2 + b$$

(a) Find the values of the constants a and b

[2 marks]

(b) In the space provided below, sketch the graph of

$$y = x^2 + 2x + 3$$

Indicate clearly the coordinates of any intersections with the coordinate axes.

[3 marks]

(c) Find the value of the discriminant of $x^2 + 2x + 3$

Explain how the sign of the discriminant relates to your sketch in part (b)

[2 marks]

The equation $x^2 + kx + 3 = 0$, where k is a constant, has no real roots.

(d) Find the set of possible values of k, giving your answer in surd form.

[4 marks]

Question 8

A-level Examination Question from January 2005, Paper C1, Q10

Given that

$$f(x) = x^2 - 6x + 18, \qquad x \geqslant 0$$

(a) express $f(x)$ in the form $(x - a)^2 + b$, where a and b are integers.

[3 marks]

The curve C with equation $y = f(x)$, $x \geqslant 0$, meets the y-axis at P and has a minimum point at Q

(b) Sketch the graph of C, showing the coordinates of P and Q

[4 marks]

The line $y = 41$ meets C at the point R

(c) Find the coordinates of R, giving your answer in the form $p + q\sqrt{2}$ where p and q are integers.

[5 marks]

Question 9

A-Level Examination Question from May 2010, Paper C1, Q10 (Edexcel)

(a) On the same axes sketch the graphs of the curves with equations

(i) $y = x(4 - x)$ **(ii)** $y = x^2(7 - x)$

showing clearly the coordinates of the points where the curves cross the coordinate axes.

[5 marks]

(b) Show that the *x*-coordinate of the points of intersection of

$y = x(4 - x)$ and $y = x^2(7 - x)$

are given by the solutions to the equation

$$x\left(x^2 - 8x + 4\right) = 0$$

[3 marks]

The point A lies on both the curve and the x and y coordinate of A are both positive.

(c) Find the exact coordinates of A, leaving your answer in the form

$$\left(p + q\sqrt{3},\ r + s\sqrt{3} \right)$$

where p, q, r and s are integers.

[7 marks]