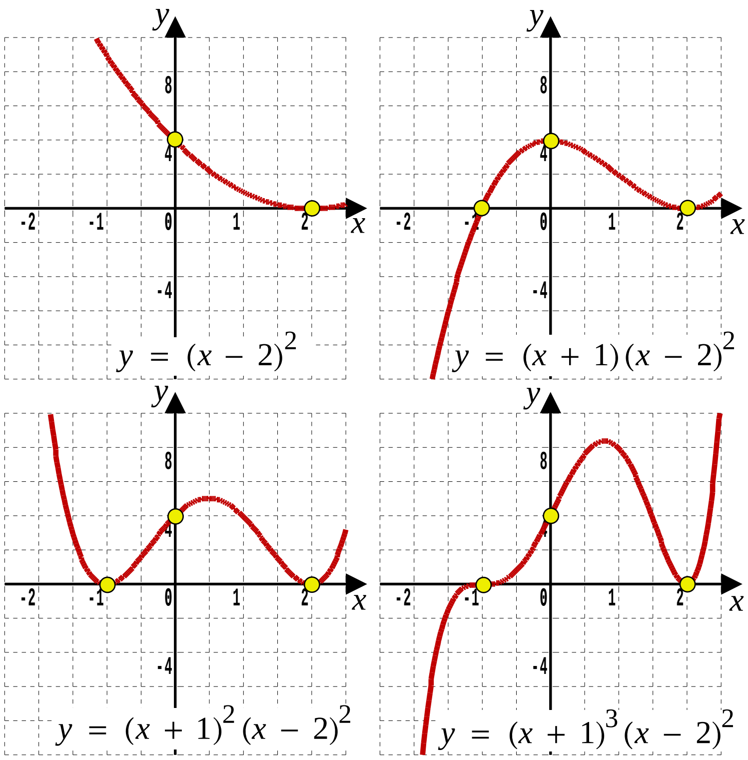
2.1 Multiplicity of Roots

Given the function, $f(x) = (x+1)^n(x-2)^2$ a mathematicain would say that that there is a root of multiplicity n at $x = -1$ and a root of multiplicity 2 at $x = 2$.
The following graphs show what happens as the multiplicity of the root at $x = -1$, n, increments from 0 to 3.
In particular, observe the activity around the number -1 on the x-axis.



Equation	n	parity of n	$x = -1$ activity	$x = 2$ activity
$y = (x-2)^2$	0	even	no root	touch from above
$y = (x+1)(x-2)^2$	1	odd	cross from below	touch from above
$y = (x+1)^2(x-2)^2$	2	even	touch from above	touch from above
$y = (x+1)^3(x-2)^2$	3	odd	cross from below	touch from above

The observations made suggest the following rule,

Cross or Touch Rule for Polynomial Roots

Given a polynomial of the form,

$$f(x) = (x - a)^n \, g(x)$$

the root at $x = a$ will be a *cross* if n is odd or a *touch* if n is even.

(Provided that $x = a$ is not also a root of the polynomial $g(x)$)

Whether the *cross* or *touch* is from above or below depends upon how the roots of $g(x)$ are distributed about $x = a$

2.2 The "Together" Sketches

Sketch each of the following curves, marking all intersections with the axes.

(i) $y = (x + 3)^3 (x - 1)^2 (x - 2)^2$

(ii) $y = x - x^3$

(iii) $y = (x + 3)^2 (4 - x)^2$

[9 marks]

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 70

Question 1

Sketch each of the following curves, marking all intersections with the axes.

(i) $y = (x)^2(x + 2)^2(x + 3)$

(ii) $y = x - x^5$

(iii) $y = (3 - x)^3(x + 2)^2$

[9 marks]

Question 2

(i) On the same axes sketch the curve $y = (x^2 - 1)(x - 2)$ and the line $y = 14x + 2$

[4 marks]

(ii) Use algebra to find the coordinates of the points of intersection.

[3 marks]

Question 3

(i) On the same axes sketch the curve with equations $y = (x - 2)(x + 2)^2$ and the curve with equation $y = -x^2 - 8$

[4 marks]

(ii) Use algebra to find the coordinates of the points of intersection.

[3 marks]

Question 4

A-Level Examination Question from January 2007, Paper C1, Q10 (Edexcel)

(a) On the same axes sketch the graphs of the curves with equations,

(i) $y = x^2(x - 2)$

(ii) $y = x(6 - x)$

and indicate on your sketches the coordinates of all points where the curves cross the *x*-axis.

[6 marks]

(b) Use algebra to find the coordinates of the points where the graphs intersect.

[7 marks]

Question 5

$$f(x) = x^3 - 2x^2 + px + 36 \text{ where } p \in \mathbb{Z}$$

(i) Given that $x = 3$ is a root of $f(x)$ determine the value of p

[3 marks]

(ii) Factorise $f(x)$ completely.

[4 marks]

(iii) Sketch the graph of $f(x)$ marking on all axis intersections.

[3 marks]

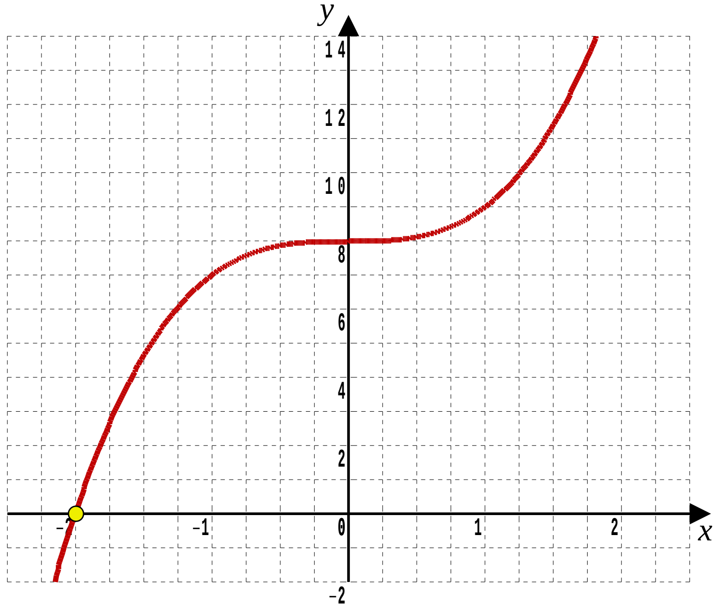
Question 6

The graph below, $y = f(x)$, is of the function $f(x) = x^3 + 8$

From the graph it is clear that $f(x)$ has only one root at $x = -2$

Mathematically, this root can be found by solving the equation $f(x) = 0$

This question looks at another mathematical way of showing there is only one root.



(i) Carry out a polynomial long division to show that,

$$x^3 + 8 = (x + 2)(x^2 + px + q)$$

where p and q are integers the values of which are to be found.

HINT: $x + 2 \overline{\big)\, x^3 + 0x^2 + 0x + 8}$

[3 marks]

(ii) By considering the discriminant of the factor $(x^2 + px + q)$ show that this quadratic factor yields no further roots of $f(x)$

[3 marks]

Question 7

$$f(x) = x^6 - 3x^5 - 15x^4 + 35x^3 + 90x^2 - 108x - 216$$

It is known that $f(x)$ has two distinct roots each of multiplicity three.

Thus, $f(x) = (x + p)^3(x + q)^3$ where $p, q \in \mathbb{Z}$ with $p < q$

(i) Explain how the factor theorem proves that neither $(x - 1)$ nor $(x + 1)$ can be factors of $f(x)$

[3 marks]

(ii) By writing 216 as a product of primes, list two candidate possible factorisations of $f(x)$ keeping in mind that $p^3 q^3 = 216$.

[3 marks]

(iii) Use the factor theorem to determine which of the two candidates factorisations is the actual factorisation of $f(x)$

State the value of p and the value of q

[3 marks]

(iv) Sketch the graph of $f(x)$ marking on all axis intersections.

[3 marks]

Question 8

$$f(x) = 3x^3 + x^2 - x \qquad \text{and} \qquad g(x) = 2x(x-1)(x+1)$$

Show algebraically that the graphs of $f(x)$ and $g(x)$ have only one point of intersection, and hence find the coordinates of this point.

[6 marks]