Lecture 5

University Undergraduate Lectures in Mathematics
A First Year Course
Group Theory II

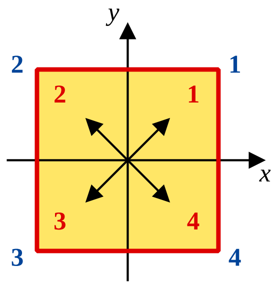
5.1 Partitions

A partition of a set X is a decomposition of the set into non-empty subsets, no two of which overlap and whose union is all of X.
This is of interest following the illustration in the previous lecture that by using $g = (1\ 2)(3\ 4)$ as a conjugating element on each of Jerome's symmetries of a square, those symmetries were partitioned as follows;

Element	e	r	r^3	x	y	r^2	p	n
Conjugate	e	r^3	r	y	x	r^2	n	p

The conjugating element, g, partitioned the symmetries of a square into the five conjugacy classes $\{e\}$, $\{r,\ r^3\}$, $\{x, y\}$, $\{r^2\}$, $\{p, n\}$.
Intuitively, these conjugacy classes correspond to "geometric type", although it's also helpful to look at the conjugacy classes by cycle structure to see the extra partitioning within "geometric type".

For example, the cycle structure for r^2 suggests the mathematics is "seeing" this symmetry as an inversion rather than a rotation.



Element	Cycle Notation	Cycle Type	Geometric Type
e	$(1)(2)(3)(4)$	$(-)(-)(-)(-)$	do nothing
r	$(1\ 2\ 3\ 4)$	$(-\ -\ -\ -)$	rotation
r^3	$(1\ 4\ 3\ 2)$	$(-\ -\ -\ -)$	rotation
x	$(1\ 4)(2\ 3)$	$(-\ -)(-\ -)$	reflection
y	$(1\ 2)(3\ 4)$	$(-\ -)(-\ -)$	reflection
r^2	$(1\ 3)(2\ 4)$	$(-\ -)(-\ -)$	inversion
p	$(2\ 4)$	$(-\ -)$	reflection
n	$(1\ 3)$	$(-\ -)$	reflection

5.2 Equivalence Relations

Definition : Relation

A relation, ~, on a set X is a rule such that, for any two elements $x, y \in X$, it is possible to determine whether x is related to y

The phrase "x is related to y", is written $x \sim y$, or $x \, R \, y$

5.2.1 Example

Let X be the infinite set of positive integers.

A relation is defined on X such that x and y are related if $hcf\,(x,\, y) \; = \; 3$

Determine which of the following are related;

(i) 6, 15 **(ii)** 12, 18 **(iii)** 24, 25

☞ Check your answer with that provided on the following page.

[3 marks]

A relation that is particularly useful from the group theoretic point of view is "an equivalence relation". In addition to being a relation, an equivalence relation has three additional properties;

Reflexive: each element is assigned to an equivalence class of "same type".

Symmetric: for elements in the class the "sameness" is mutual.

Transitive: if an element is related to one element in an equivalence class then it is related to every element in that class.

Definition : Equivalence Relation

An equivalence relation on a set X is a relation ~ on X which satisfies the following three axioms:

Reflexive: For all $x \in X$, $x \sim x$

Symmetric: For all $x, y \in X$, if $x \sim y$ then $y \sim x$

Transitive: For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$

5.2.2 Example

Let X be the set of positive integers.

Prove that the relation "has parity with" is an equivalence relation.

[3 marks]

☞ Check your answer with that provided on the following page.

5.3 Answers to 5.2 Examples

5.3.1 Solution to 5.2.1 Example

(i) $hcf\,(6,\ 15) = 3$ $\therefore$ 6 is related to 15, i.e. $6 \sim 15$

(ii) $hcf\,(12,\ 18) = 6$ $\therefore$ 12 is not related to 18

(iii) $hcf\,(24,\ 25) = 1$ $\therefore$ 24 is not related to 25

[3 marks]

5.3.2 Solution to 5.2.2 Example

The relation "has parity with" is referring to the property of a positive integer being either an odd or an even number.
To show this is an equivalence relation the three axioms need to be checked.

Reflexive: An even number has parity with itself as does an odd number.
$\therefore$ for all $x \in X$, $x \sim x$
The relation is reflexive.

Symmetric: Given two numbers if the first has parity with the second, then the second has parity with the first.
$\therefore$ for all $x, y \in X$, if $x \sim y$ then $y \sim x$
The relation is symmetric.

Transitive: Given three numbers, if the first has parity with the second, and the second has parity with the third, then the first has parity with the third.
$\therefore$ for all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$
The relation is transitive.

Hence "has parity with", ~, is an equivalence relation. □

The above proof is not as rigorous as it could be. However, the 5.7 Exercise will give practice at giving a more robust demonstration that each of the axioms has been satisfied once a more detained proof has been studied (in section 5.5).

[3 marks]

5.4 Equivalence Class

Definition: Equivalence Class
Let ~ be an equivalence relation defined on a set X, in which case the equivalence class of $x \in X$, denoted by $[x]$, is the set,

$$[x] = \{y \in X : x \sim y\}$$

- In practice, if the context is clear, the square brackets may be dropped.
- In the earlier example on parity, all of the even integers were in one equivalence class and all of the odd integers in another.

5.5 "Conjugate to" is an Equivalence Relation

The "Conjugate to" Theorem
In any group G, the relation "conjugate to" is an equivalence relation on the set of elements of G.

The teaching video (link on the next page) will talk through the following proof that "conjugate to" is an equivalence relation.

Proof : The three axioms that define an equivalence relation need to each be considered in turn.

Reflexive: For all $x \in G$, it is true that $x = e\,x\,e^{-1}$
Thus e is a conjugating element that conjugates each element x to itself and so $x \sim x$
The relation is reflexive.

Symmetric: Given that $x, y \in G$ and that $x \sim y$ then there exists a conjugating element $g \in G$ such that,

$y = g\,x\,g^{-1}$	"x is conjugate to y"
$g^{-1}y = g^{-1}g\,x\,g^{-1}$	Pre-multiply both sides by g^{-1}
$g^{-1}y = x\,g^{-1}$	As $g^{-1}g = e$
$g^{-1}y\,g = x\,g^{-1}g$	Post-multiply both sides by g
$g^{-1}y\,g = x$	As $g^{-1}g = e$
$x = g^{-1}y\,g$	But $g = \left(g^{-1}\right)^{-1}$
$x = \left(g^{-1}\right) y \left(g^{-1}\right)^{-1}$	"y is conjugate to x"

Which shows that g^{-1} is the conjugating element that conjugates y to x i.e. $y \sim x$
The relation is symmetric.

Transitive: Given that $x, y, z \in G$ and that $x \sim y$ and $y \sim z$ then there exist elements $g, h \in G$ such that,

$y = g\,x\,g^{-1}$ and $z = h\,y\,h^{-1}$

$z = h\,g\,x\,g^{-1}h^{-1}$	Substitution
$= (hg)\,x\left(g^{-1}h^{-1}\right)$	Associativity
$= (hg)\;x\;(gh)^{-1}$	Proven Lecture 1, Q3

Which shows that $h\,g$ is the conjugating element that conjugates x to z That is, $x \sim z$
The relation is transitive.

Hence "conjugate to", ~, is an equivalence relation. □

5.6 Teaching Video for 5.5 Proof

Teaching Video: http://www.NumberWonder.co.uk/v9110/5.mp4

5.7 Exercise

Marks Available: 50

Question 1

Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
An equivalence relation is defined on G such that $x \sim y$ if x has the same number of factors as y. Here is a partially completed table which shows how G is partitioned into the various equivalence classes;

Equivalence Class	Elements of G in class
$[1]$	$\{1\}$
$[2]$	$\{2, 3, 5, 7, 11, 13\}$
$[3]$	
$[4]$	
$[5]$	
$[6]$	

(i) Complete the table

[3 marks]

(ii) Describe the equivalence class $[2]$ in words as simply as possible.

[1 marks]

(iii) If G were extended to be the set of all positive integers, in which equivalence class or classes wouldn't any square number belong ?

[2 marks]

Question 2

A relation $x \sim y$ is defined on the set of real numbers as,

$$x \sim y \iff (x - y)^2 (xy - 1)^2 = 1$$

(i) Show that $2 \sim 1$ satisfies this relation.

[1 mark]

(ii) Show that $1 \sim 0$ satisfies this relation.

[1 mark]

(iii) Given $x, y, z \in G$, $x \sim y$ and $y \sim z$, a relation is transitive if $x \sim z$.
With $x = 2$, $y = 1$ and $z = 0$ show that the relation is not transitive.

[1 mark]

(iv) Is this relation symmetric ?
Justify your answer.

[2 marks]

(v) Is this relation reflexive ?
Justify your answer.

[2 marks]

(vi) Is the relation an equivalence relation ?
Justify your answer.

[1 mark]

Question 3

Given that $X = \{1, 2, 3, 4, 5, 6, 7\}$ which of the following two options is a partition of A that will give rise to an equivalence relation.
Justify your answer.

Option A : $A_1 = \{1, 3, 5\}$, $A_2 = \{2\}$, $A_3 = \{4, 7\}$

Option B : $B_1 = \{1, 2, 5, 7\}$, $B_2 = \{3\}$, $B_3 = \{4, 6\}$

[2 marks]

Question 4

A relation is defined on the set of integers as,

$$x \sim y \text{ if } x + 3y \text{ is a multiple of 4}$$

(i) Show that this relation is an equivalence relation by giving mathematical proofs that each of the reflexive, symmetric and transitive axioms hold.

[8 marks]

Question 5

A relation is defined on the set of integers as,

$$x \sim y \quad \text{if} \quad 3x^2 - y^2 \ \text{ is divisible by 2}$$

(i) Show that this relation is an equivalence relation by giving mathematical proofs that each of the reflexive, symmetric and transitive axioms hold.

[8 marks]

(ii) State the number of distinct equivalence classes, and describe each class in words as simply as possible.

[2 marks]

Question 6
A relation is defined on the set of real numbers as,

$$x \sim y \quad \text{if} \quad |x - y| \leqslant 3$$

Determine, with proofs, whether each of the reflexive, symmetric and transitive axioms hold and hence if this relation is an equivalence relation.

[6 marks]

Question 7

Abelian Group's Singletons Theorem
In an Abelian group, each conjugacy class comprises a single element.

Prove the Abelian Group's Singletons Theorem

[4 marks]

Question 8

Conjugate Powers Theorem

If g conjugates x to y then g conjugates x^n to y^n for any integer n.

In other words, if $y = g\,x\,g^{-1}$ then $y^n = g\,x^n\,g^{-1}$ for $\mathbb{Z}$

Use mathematical induction to prove the above theorem for positive integers.

[6 marks]