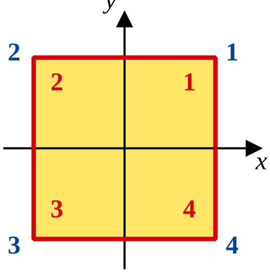
Lecture 4

University Undergraduate Lectures in Mathematics
A First Year Course
Group Theory II

4.1 The Shuffled Square

Two mathematicians, Jerome and Emily, are labelling the vertices of a square ahead of setting up the cycle notation for the three symmetries; rotation of 90°, reflection in the line $y = x$ and reflection in the x-axis.
Here, side by side, are their proofs that the 90° rotation followed by a reflection in the line $y = x$ is equivalent to a reflection in the x-axis.
Notice that, right at the start, they label their squares differently.

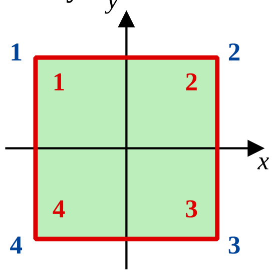
Jerome



From my diagram;

$$r = (1\,2\,3\,4)$$
$$p = (2\,4)$$
$$x = (1\,4)(2\,3)$$
$$\text{LHS} = p \circ r$$
$$= (2\,4) \circ (1\,2\,3\,4$$
$$= (1\,4)(2\,3)$$
$$= x$$
$$= \text{RHS} \quad \square$$

Emily



From my diagram;

$$R = (1\,4\,3\,2)$$
$$P = (1\,3)$$
$$X = (1\,4)(2\,3)$$
$$\text{LHS} = P \circ R$$
$$= (1\,3) \circ (1\,4\,3\,2)$$
$$= (1\,4)(2\,3)$$
$$= X$$
$$= \text{RHS} \quad \square$$

The group of symmetries associated with Jerome's square is clearly isomorphic to the group of symmetries of Emily's square. The isomorphism is simply,

Jerome's Label	ϕ $\rightarrow$	Emily's Label
1	$\rightarrow$	2
2	$\rightarrow$	1
3	$\rightarrow$	4
4	$\rightarrow$	3

Notice that, $\phi = (1\,2)(3\,4)$ but the operation "relabel the vertices" is NOT the same as the square's binary operation of symmetry compositions.
So, for example, writing $\phi \circ r = R$ is not true and, in fact, makes no sense at all.

However, one could write: $r = (1\,2\,3\,4) \xrightarrow{\phi} (2\,1\,4\,3) = (1\,4\,3\,2) = R$

Here is a table of all of Jerome's and all of Emily's symmetries;

For Jerome's Square			For Emily's Square	
e	$(1)(2)(3)(4)$		E	$(1)(2)(3)(4)$
r	$(1\ 2\ 3\ 4)$		R	$(1\ 4\ 3\ 2)$
r^2	$(1\ 3)(2\ 4)$		R^2	$(1\ 3)(2\ 4)$
r^3	$(1\ 4\ 3\ 2)$		R^3	$(1\ 2\ 3\ 4)$
x	$(1\ 4)(2\ 3)$		X	$(1\ 4)(2\ 3)$
y	$(1\ 2)(3\ 4)$		Y	$(1\ 2)(3\ 4)$
p	$(2\ 4)$		P	$(1\ 3)$
n	$(1\ 3)$		N	$(2\ 4)$

The eight symmetries of a square are a subgroup of S_4. The order of S_4 is 24 and yet, in spite of this, Jerome and Emily have exactly the same collection of eight permutations for their symmetries. Why has this happened ?

It's because the relabelling function, ϕ, in this case, happened to be the same permutation as one of the elements of the group; In fact, Jerome's square when reflected in the y-axis gives Emily's square.
The group property of closure then guarantees the same eight expressions when written in cycle notation will be in both Jerome's and Emily's symmetry list, but shuffled around.

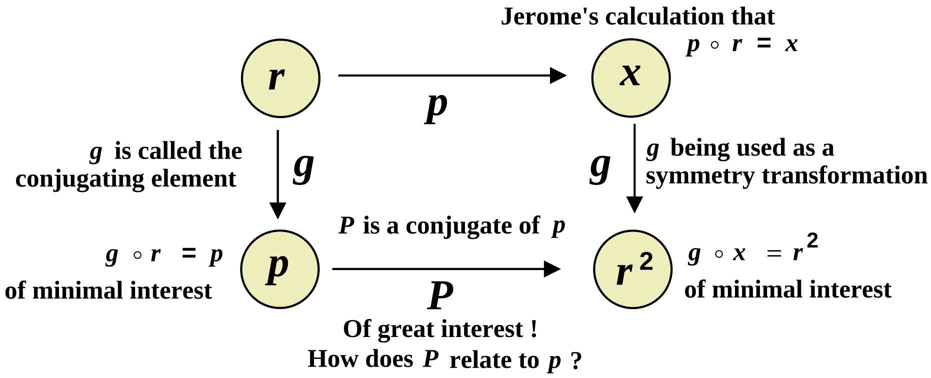
In consequence, the isomorphism between these two groups is, effectively, an isomorphism from the group back onto itself.

Definition : Automorphism
An automorphism of a group G is an isomorphism from G to G

- Not all possible relabellings of the square result in an automorphism.
  (This fact will be demonstrated in this lecture's exercises)

When a relabelled automorphism, ϕ, is used as if it were a symmetry composition it is called a conjugating element and, as it's an element from G, is often denoted g.

A diagram can now be drawn to show the effect g is having when treated, not as a renaming tool, but as a symmetry of the square. That is, a reflection in the, y-axis.



By traversing the diagram from r to r^2 by going both anticlockwise and clockwise and equating the result a relationship between P and p can be established,

$P \circ g \circ r = r^2$ by working anticlockwise from r to r^2

$g \circ p \circ r = r^2$ by working clockwise from r to r^2

$$\therefore \; P \circ g \circ r = g \circ p \circ r$$

$$P \circ g = g \circ p$$

$$P = g \circ p \circ g^{-1}$$

For Jerome's square, $P = (1\,2)(3\,4) \circ (2\,4) \circ (1\,2)(3\,4)$

$= (1\,3)$ note $g = g^{-1}$ in this case

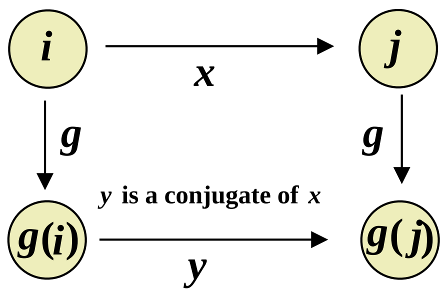
$= n$ $\therefore$ n is a conjugate of p

4.2 Generalising

Conjugate Elements

Let x and y be elements of a group G; then y is a conjugate of x in G if there exists an element $g \in G$ such that,

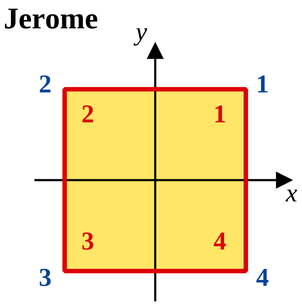
$$y = g x g^{-1}$$



Note: i, j, $g(i)$, $g(j)$ are element of G but exactly which elements they are is unimportant when finding conjugates.

4.3 Example

With $g = (1\ 2)(3\ 4)$ as conjugating element determine the element that is the conjugate of $r = (1\ 2\ 3\ 4)$ in Jerome's group of symmetries of a square.



Teaching Video: http://www.NumberWonder.co.uk/v9110/4.mp4

[2 marks]

4.4 Conjugacy Classes

Using $g = (1\ 2)(3\ 4)$ as the congugating element with each of the remaining elements of Jerome's group of symmetries of a square leads to the following;

Element	e	r	r^3	x	y	r^2	p	n
Conjugate	e	r^3	r	y	x	r^2	n	p

The congugating element, g, has partitioned the elements into the five conjugacy classes $\{e\}$, $\{r,\ r^3\}$, $\{x, y\}$, $\{r^2\}$, $\{p, n\}$

4.5 Cycle Structure Preserved

Conjugacy preserves cycle structure, which is intuitively obvious if it is recalled that an interpretation of what the conjugating element does in any permutation group is “rename”.

To be clear, conjugate elements must have the same cycle structure.
In the table above, for example both *x* and *y* have the cycle structure (- -)(- -)

As cycle structure determines order, a consequence of cycle structure being preserved is that conjugate elements must also have the same order.

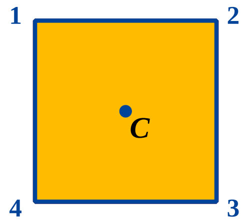
4.6 Exercise

Marks Available: 40

Question 1

University Mock Examination Question from 1995, M203, Q9 (OU)

Consider G, the symmetry group of the square below, with centre C.



Let $h \in G$ be the anticlockwise rotation through $\frac{\pi}{2}$ about the centre C, and let $g \in G$ be the reflection in the vertical axis through C.

(a) Write down h and g in cycle form, using numbering of the locations of the vertices as shown.

[2 marks]

(b) Write down the conjugate $g\,h\,g^{-1}$ of h, and state its order.

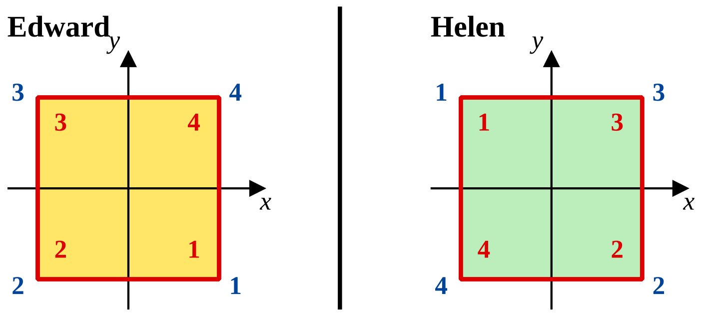
[2 marks]

(c) State whether or not g and h are conjugate, giving a brief reason for your answer.

[2 marks]

Question 2

Two mathematicians, Edward and Helen, are labelling the vertices of a square ahead of working out the symmetries of their squares in cycle notation.
Notice that, right at the start, they labelled their squares differently.



(i) Complete the following table where,

- e, E : identity elements
- r, R : rotation of 90° about the square's centre
- x, X : reflection in the x-axis
- y, Y : reflection in the y-axis
- p, P : reflection in the line $y = x$
- n, N : reflection in the line $y = -x$

For Edward's Square			For Helen's Square	
e	(1)(2)(3)(4)		E	(1)(2)(3)(4)
r			R	
r^2			R^2	
r^3			R^3	
x			X	
y			Y	
p			P	
n			N	

[7 marks]

(ii) Explain why the relationship between Edward's and Helen's groups of symmetries of the square could be considered to be an isomorphism but not an automorphism.

[2 marks]

Question 3

University Assessment Question from 1996, M203, Q5(1) (OU)

(a) Find all elements in S_6 that conjugate $(1\ 5\ 3\ 6\ 4\ 2)$ to itself.

[5 marks]

(b) For each element found in part (a), give its order and parity.

[5 marks]

(c) Prove directly (verify that the subgroup axioms hold) that the set of elements that you found in part (a) is a subgroup of S_6 and identify this subgroup up to isomorphism. (In other words, identify a group of symmetries of some figure to which your subgroup is isomorphic)

[6 marks]

(d) Find all the elements in S_6 that conjugate $(1\ 5\ 3\ 6\ 4\ 2)$ to $(1\ 2\ 3\ 4\ 5\ 6)$

[5 marks]

(e) Chose one element, p, that conjugates $(1\ 5\ 3\ 6\ 4\ 2)$ to $(1\ 2\ 3\ 4\ 5\ 6)$ and show that the elements $p \circ x$, where x conjugates $(1\ 5\ 3\ 6\ 4\ 2)$ to itself (from part (a)), are precisely the elements found in part (d).

[4 marks]