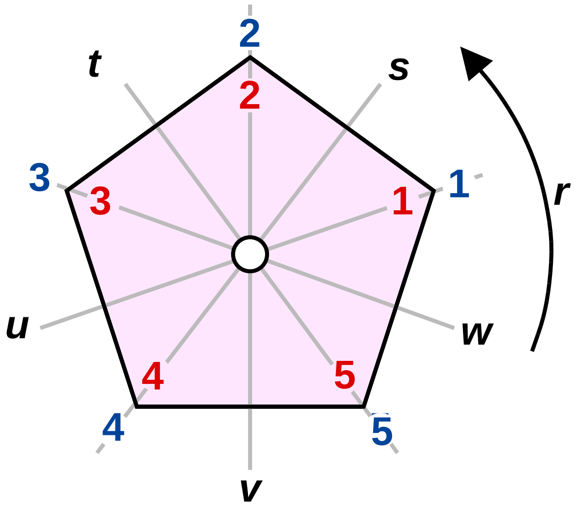
2.1 Cycle Notation

To study the symmetries of a regular pentagon, the five vertices are numbered from 1 to 5. The five lines of mirror symmetry are labelled s, t, u, v and w.
A rotation of 72° is denoted r, as shown in the diagram.



The set G is $\{e,\ r,\ r^2, r^3,\ r^4, s, t, u, v, w\}$, where the elements represent transformations. The identity is e and r^2, for example, is a rotation of 144°.
Under the binary operation "followed by" the group $(G, \circ)$ of symmetries of a regular pentagon is established.

Under the transformation r^2 the red vertices move into these blue positions:

$$1 \to 3,\quad 2 \to 4,\quad 3 \to 5,\quad 4 \to 1,\quad 5 \to 2$$

This can be expressed in two line permutation notation as,

$$r^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

Similarly, the reflection in s can be written,

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

However, this notation is cumbersome; the red line is always the same and how the shape actually moved is not clear at a glance.
In an improved notation, called "cycle notation" the movement instructions are lined up differently;

$$1 \to 3,\quad 3 \to 5,\quad 5 \to 2,\ 2 \to 4,\quad 4 \to 1$$

The idea is to group like so;

$$1 \to (\ 3, 3\ ) \to (\ 5, 5\ ) \to (\ 2, 2) \to (\ 4, 4\ ) \to 1$$

before throwing away the unnecessary blue doubles and commas;

$$r^2 = (1\ 3\ 5\ 2\ 4)$$

There is no necessity to start the cycle from 1 with the result that there are the following equivalent ways of describing the rotation of 144°;

$$r^2 = (1\,3\,5\,2\,4) = (3\,5\,2\,4\,1) = (5\,2\,4\,1\,3) = (2\,4\,1\,3\,5) = (4\,1\,3\,5\,2)$$

Not all of the pentagon's symmetry permutations cycle though all five digits as was the case in the above example.
To see this, consider the reflection *s*.
Here is the logic behind how that is written in cycle notation;

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

$$1 \to 2, \quad 2 \to 1, \; 3 \to 5, \quad 4 \to 4, \quad 5 \to 3$$

$$1 \to 2, \quad 2 \to 1 \quad : \quad 3 \to 5, \quad 5 \to 3, \quad : \quad 4 \to 4$$

$$1 \to (\,2, 2\,) \to 1 \quad : \quad 3 \to (\,5, 5\,) \to 3 \quad : \quad (\,4, 4\,)$$

$$s = (1\,2)(3\,5)(4)$$

By convention, any point that is fixed, such as in this case the 4, may be dropped.

$$\therefore \; s = (1\,2)(3\,5)$$

Again this can be written in a variety of equivalent ways;

$$s = (1\,2)(3\,5) = (1\,2)(5\,3) = (2\,1)(3\,5) = (2\,1)(5\,3)$$

$$= (3\,5)(1\,2) = (3\,5)(2\,1) = (5\,3)(1\,2) = (5\,3)(2\,1)$$

2.2 Composition

By continuing to work with the symmetry group of the pentagon and working in cycle notation, show that $s\,r^2 = w$

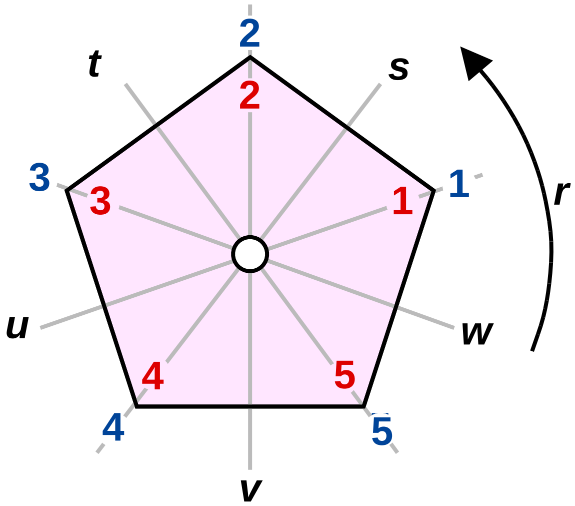
Teaching video: http://www.NumberWonder.co.uk/v9110/2.mp4

[4 marks]

2.3 Exercise

Marks Available: 40

Question 1



Complete the following table to show each of the symmetries of the pentagon written in cycle notation;

Symmetry	Cycle Notation
e	(1)(2)(3)(4)(5)
r	
r^2	(1 3 5 2 4)
r^3	
r^4	
s	(1 2) (3 5)
t	
u	
v	
w	(1 5) (2 4)

[5 marks]

Question 2

Show that $s\,r^2 \neq r^2\,s$

[2 marks]

Question 3

Complete the Cayley table for the group of symmetries of a regular pentagon.

$*$	e	r	r^2	r^3	r^4	s	t	u	v	w
e	e	r	r^2	r^3	r^4			u	v	w
r	r	r^2	r^3	r^4	e			s	t	u
r^2	r^2	r^3	r^4	e	r			v	w	s
r^3	r^3	r^4	e	r	r^2			t	u	v
r^4	r^4	e	r	r^2	r^3			w	s	t
s										
t										
u	u	w	t	v	s			e	r^3	r
v	v	s	u	w	t			r^2	e	r^3
w	w	t	v	s	u			r^4	r^2	e

[6 marks]

Question 4

A transposition written in cycle notation is a permutation amongst n elements which swaps exactly two of them.

For example, when working with the group of symmetries of the pentagon, the transposition $(1\ 4)$ can be invoked. This is in spite of the fact that there is no symmetry of the pentagon that swaps only the 1 with the 4, leaving all else fixed. That is, there was no pentagon symmetry that had the two-line permutation,

$$\chi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$$

(i) Verify that for the regular pentagon,

$$(1\ 2) \circ (3\ 1) \circ (4\ 1) \circ (1\ 5) = r^4$$

[2 marks]

(ii) To which symmetry of the pentagon is the following transposition equivalent ?

$$(2\ 4) \circ (1\ 2) \circ (2\ 3) \circ (5\ 2)$$

[2 marks]

(iii) To which symmetry of the pentagon is the following transposition equivalent ?

$$(5\ 3) \circ (3\ 2) \circ (4\ 3) \circ (1\ 3)$$

[2 marks]

Question 5
Any permutation can be decomposed into a composition of transpositions.
The decomposition is not unique (As was demonstrated by Question 4)
What is unique is whether the result has an even or an odd number of transpositions.

Permutation Decomposition Technique

$$(g_1\, g_2\, g_3\, \dots\, g_n) = (g_1\, g_n) \circ (g_1\, g_{n-1}) \circ \dots \circ (g_1\, g_3) \circ (g_1\, g_2)$$

The symmetry of the pentagon, r^2, can be written is cycle notation in several ways including $(1\;3\;5\;2\;4)$ or $(5\;2\;4\;1\;3)$

Use the Permutation Decomposition Technique to write the following as a composition of transpositions;

(i) $r^2 = (1\;3\;5\;2\;4)$

[2 marks]

(ii) $r^2 = (5\;2\;4\;1\;3)$

[2 marks]

(iii) In both part (i) and part (ii) is the parity of the composition of transpositions odd or even ?

[1 mark]

There is a theorem, called the Parity Theorem, which confirms what the above suggests;

The Parity Theorem
A permutation cannot be written both as a composite of an even number of transpositions and as a composite of an odd number of transpositions.
In other words, the properties of being odd or being even are mutually exclusive for compositions of transpositions.

Question 6

Open University Examination Question, (M203) (edited)

The two-line symbols for permutations p and q in the permutation group S_6 are

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 6 & 5 \end{pmatrix} \text{ and } q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$

(a) Write down the permutations $p,\ q,\ p^{-1}$ and q^{-1} in cycle form.

[4 marks]

(b) Determine the products $p \circ q$ and $q \circ p$ in cycle notation.

[2 marks]

(c) Write p and q as products of transpositions.
State the parity of p and q.

[4 marks]

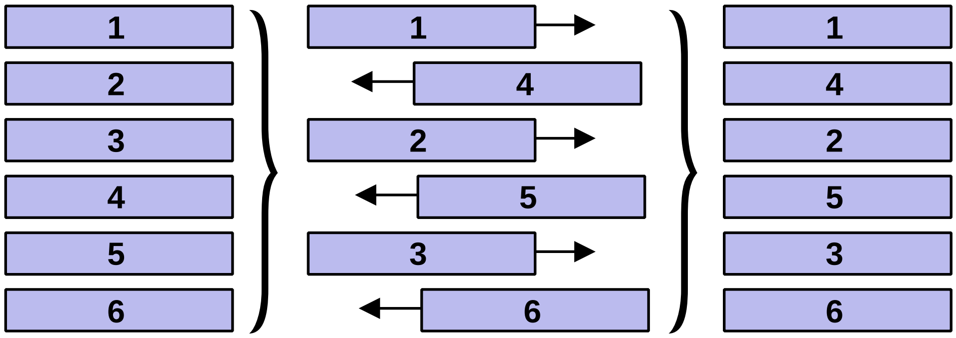
(d) Let r be the permutation in S_6 that is given in cycle form as $(2\ 3\ 4\ 5)$

Find a permutation s in S_6 such that $r \circ s = q \circ r$

[2 marks]

Question 7

A deck of six cards is cut exactly in half and a perfect riffle shuffle carried out. The diagram below shows this Faro shuffle.



(i) Write down in cycle notation the permutation of one such shuffle.

[1 mark]

(ii) Call the part (i) permutation *s*.

Calculate s^2, s^3 and so on until the identify permutation is reached.

The identity permutation could be written $e = (1)(2)(3)(4)(5)(6)$

[2 marks]

(iii) Explain the significance of your part (ii) answer in regards to the repeated perfect Faro riffle shuffling of the deck of six cards.

[1 mark]