“Doctor, doctor, I keep seeing into the future”
“When did this start ?”
“Next Tuesday”

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

The line L has equation $r = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$

Identify which **one** of the following lines is perpendicular to L and prove your claim.

$r_1 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ $\qquad$ $r_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ $\qquad$ $r_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \mu_3 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

[3 marks]

Question 2

Complex Numbers Division Rule

For any two complex numbers z_1 and z_2

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

This result may be quoted in examinations; worth knowing !

(i) Prove this result by by letting
- $z_1 = r_1 (cos\, \theta_1 + \text{i}\, sin\, \theta_1)$
- $z_2 = r_2 (cos\, \theta_2 + \text{i}\, sin\, \theta_2)$

and working out $\frac{z_1}{z_2}$

[5 marks]

(ii) If $z_1 = 1 - \sqrt{3}\ \text{i}$ and $z_2 = \sqrt{3} + \text{i}$ determine,

(a) $arg(z_1)$

[1 mark]

(b) $arg(z_2)$

[1 mark]

(c) $arg\left(\frac{z_1}{z_2}\right)$

[1 mark]

Question 3

(i) Use the facts that $cosh\,x = \dfrac{e^{x} + e^{-x}}{2}$ and $sinh\,x = \dfrac{e^{x} - e^{-x}}{2}$

to prove that $tanh\,x = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

[3 marks]

(ii) As x becomes large and positive, what value does $tanh\,x$ tend towards ?
Justify your answer.

[2 marks]

(iii) As x becomes large and negative, what value does $tanh\,x$ tend towards ?
Justify your answer.

[2 marks]

(iv) Hence, sketch the graph of $y = tanh\,x$

[2 marks]

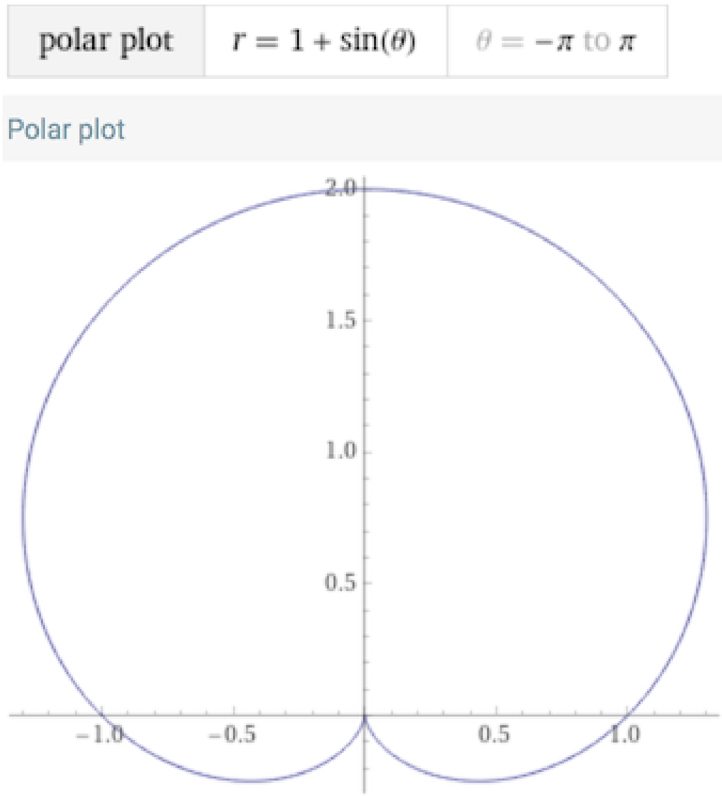
(v) Given that $3\,sinh\,x = p\,cosh\,x$ has real solutions, determine the range of possible values for p

[2 marks]

Question 4

The graph is a Wolfram Alpha™ plot of the polar equation

$$r = 1 + \sin\theta \text{ over the interval } -\pi \leqslant \theta \leqslant \pi$$



Use differentiation as part of a proof that shows that there could be a vertical tangent to the curve that touches the curve at the point with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$

[6 marks]

Question 5

(i) Given that $y = arccos\,x$ over the interval $-1 \leqslant x \leqslant 1$, show that,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

If your solution involves the taking of square roots, explain carefully why you select the negative option and discard the positive, if appropriate.

[4 marks]

(ii) $f(x) = arccos\left(e^x\right) \qquad x \leqslant 0$

(a) Explain why domain restriction $x \leqslant 0$ is necessary.

[2 marks]

(b) Prove that $f(x)$ has no stationary points.

[2 marks]

Question 6

(i) Express as partial fractions; $\dfrac{4x^2 + 3x + 14}{(x + 2)(x^2 + 4)}$

[3 marks]

(ii) Hence find the exact value of $\displaystyle\int_0^1 \frac{4x^2 + 3x + 14}{(x + 2)(x^2 + 4)}\,dx$

Simplify your answer.

[5 marks]

Question 7

After you have done this question you may like to compare the method this old examination question guides you towards with that used in question 2.

$$z_1 = 4 + 2\text{i} \qquad z_2 = -3 + \text{i}$$

(i) Find the exact value of $| z_1 - z_2 |$

[2 marks]

Given that $w = \dfrac{z_1}{z_2}$

(ii) express w in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$

[2 marks]

(iii) find *arg w*, giving your answer in radians.

[2 marks]