Lesson 7

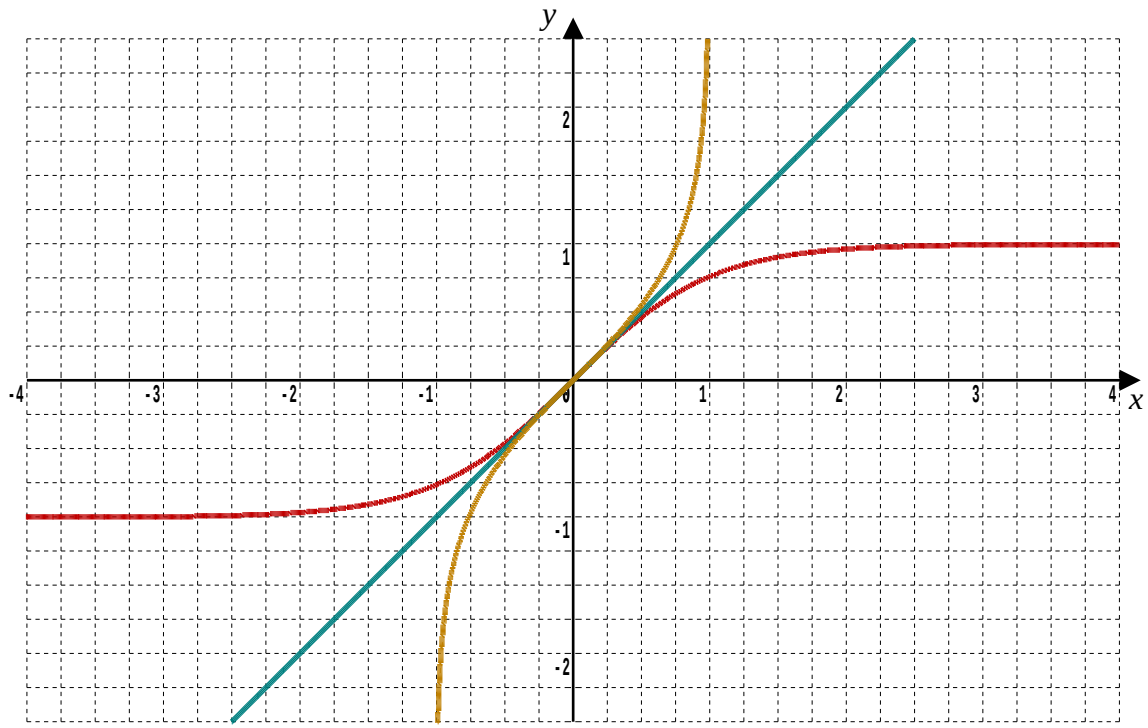
Further A-Level Pure Mathematics, Core 2

Hyperbolic Functions

7.1 *artanh x*

Like *sinh x*, the function *tanh x* is one-to-one on an unrestricted domain although, unlike *sinh x* it has horizontal asymptotes at $y = \pm 1$. Like any one-to-one function, it has an inverse that, graphically, is a reflection in $y = x$.

The inverse of *tanh x* is called *artanh x*.



Red : $y = \tanh x$ Blue : $y = x$ Gold : $y = \operatorname{artanh} x$

Like *sinh x* and *cosh x*, the *tanh x* function is defined in terms of exponentials, and its inverse, *artanh x*, involves logarithms.

The Inverse Of *tanh x* : *artanh x*

$$\operatorname{artanh} x = \frac{1}{2} ln\left(\frac{1 + x}{1 - x}\right) \qquad x \in \mathbb{R},\ |x| < 1$$

A proof of this is written out on the next page.

An excellent video from *Exam Solutions* talks through the proof

Teaching Video: http://www.NumberWonder.co.uk/v9102/7.mp4

7.2 The Proof

$$
\begin{aligned}
y &= artanh\, x \\
\therefore x &= tanh\, y \\
&= \frac{sinh\, y}{cosh\, y} \\
&= \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y} \\
&= \frac{(e^y)^2 - 1}{(e^y)^2 + 1} \\
x(e^y)^2 + x &= (e^y)^2 - 1 \\
1 + x &= (e^y)^2 - x(e^y)^2 \\
1 + x &= (e^y)^2(1 - x) \\
e^y &= \pm\sqrt{\frac{1 + x}{1 - x}}
\end{aligned}
$$

Now, since $e^y > 0$, $e^y = \sqrt{\frac{1 + x}{1 - x}}$

$$
\begin{aligned}
&= \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}} \\
y &= ln\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}
\end{aligned}
$$

That is, $artanh\, x = \frac{1}{2}\, ln\left(\frac{1 + x}{1 - x}\right)$ □

The domain of the inverse function is the range of the original function

That is $x \in \mathbb{R}$, $|x| < 1$

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 35

Question 1

Further A-Level Examination Question from June 2018, FP3, Q1 (Edexcel)

Solve the equation

$$15\,sech^2 x + 7\,tanh\,x = 13$$

Give your answers in terms of simplified natural logarithms

[6 marks]

Question 2

Further A-Level Examination Question from June 2009, FP3, Q1 (Edexcel)

Solve the equation

$$7\,sech\,x - tanh\,x = 5$$

Give your answers in the form *ln a* where *a* is a rational number

[5 marks]

Question 3

Find the equation of the tangent at the point where $x = \frac{12}{13}$ on the curve with equation $y = artanh\,x$

[3 marks]

Question 4

Further A-Level Examination Question from June 2006, FP2, Q5 (Edexcel)

The curve with equation

$$y = -x + \tanh(4x) \qquad x \geqslant 0$$

has a maximum turning point A

(a) Find, in exact logarithmic form, the x-coordinate of A

[4 marks]

(b) Show that the y-coordinate of A is $\frac{1}{4}\left\{ 2\sqrt{3} - \ln\left(2 + \sqrt{3} \right) \right\}$

[3 marks]

Question 5

Given that,

$$artanh\, x + artanh\, y = ln\sqrt{3}$$

prove that

$$y = \frac{2x - 1}{x - 2}$$

[6 marks]

Question 6

Further A-Level Examination Question from June 2008, FP2, Q1 (Edexcel)

Show that $\frac{d}{dx}\left[\ln\left(\tanh x \right) \right] = 2\,\mathrm{csch}(2x) \qquad x > 0$

[4 marks]

Question 7

Further A-Level Examination Question from June 2004, P5, Q1(b) (Edexcel)

Solve $\mathrm{csch}\, x - 2\coth x = 2$ giving your answer in the form $k \ln a$ where k and a are integers

[4 marks]