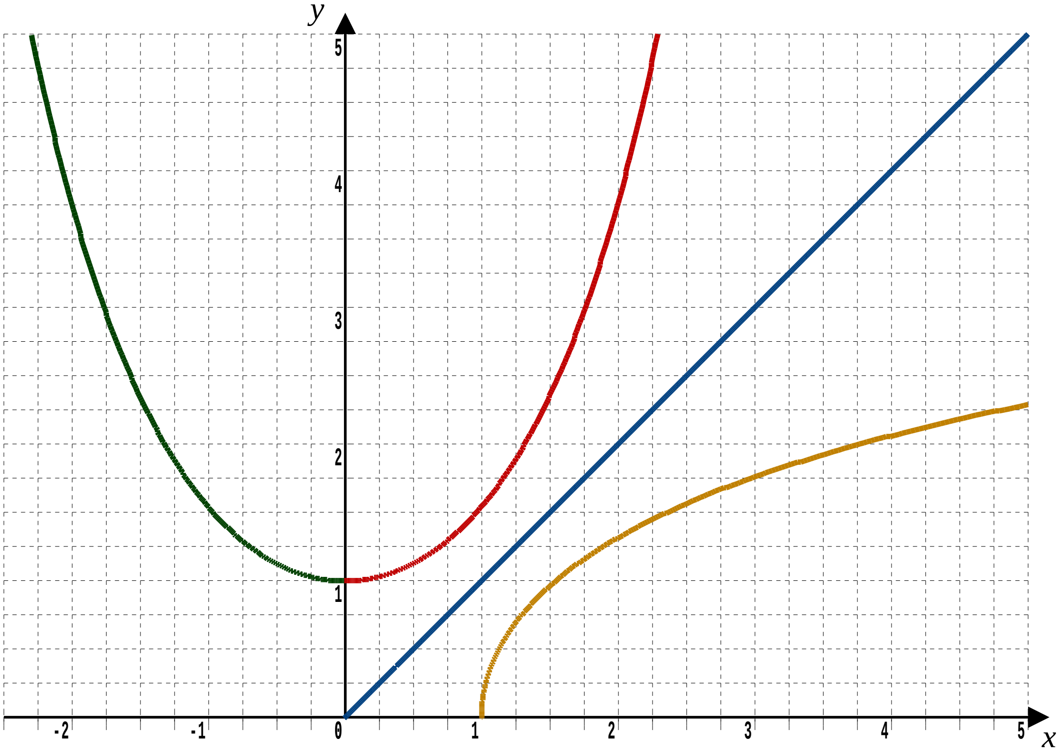
5.1 *arcosh x*

Before the inverse of the *cosh* function can be found, the domain must be restricted to obtain a one-to-one function. On the graph below the curve $y = \cosh x$ is coloured green and red, the green part being excluded by the restriction on the domain $x \geqslant 0$. The red part is then reflected in the line $y = x$ to obtain $y = \text{arcosh}\, x$, coloured gold.



The Inverse Of *cosh x* : *arcosh x*

$$\text{arcosh}\, x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \in \mathbb{R},\ x \geqslant 1$$

A proof is on the next page and talked through in the teaching video from *Exam Solutions*. It's similar to the *arsinh* proof but with a few extra technicalities.

Teaching Video: http://www.NumberWonder.co.uk/v9102/5.mp4

5.2 The Proof

$$y = arcosh\, x$$

$$\therefore x = cosh\, y$$

$$= \frac{e^y + e^{-y}}{2} \qquad \text{From the definition of } cosh$$

$$2x = e^y + e^{-y}$$

$$2x\, e^y = (e^y)^2 + 1 \qquad \text{From multiplying through by } e^y$$

$$(e^y)^2 - 2x(e^y) + 1 = 0 \qquad \text{Which is a "quadratic in disguise"}$$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm \sqrt{4}\sqrt{x^2 - 1}}{2}$$

$$= x \pm \sqrt{x^2 - 1}$$

$$y = ln\left(x \pm \sqrt{x^2 - 1}\right)$$

Now, as the *arcosh* function is one-to-one only one of these two possibilities can apply. The one to select, corresponding to the gold curve that in turn came from the red half of the *cosh* curve is the following,

$$arcosh\, x = ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \in \mathbb{R},\ x \geqslant 1 \qquad \square$$

(The other solution would correspond to the inverse of the green half of the *cosh* curve, and would be a reflection of the gold curve in the *x*-axis)

5.3 Example

(i) Determine the exact value of *arcosh*(4)

(ii) Solve, to three decimal places, $cosh\, w = 2$

[3 marks]

Solution :

(i) $arcosh(4) = ln\left(4 + \sqrt{15}\right)$ About 2.06

(ii) $w = arcosh(2)$ i.e. $x = 2$

$$= ln\left(x \pm \sqrt{x^2 - 1}\right)$$

$$= ln\left(2 \pm \sqrt{3}\right)$$

$$= \pm 1.316$$

This example is also covered in the *Exam Solutions* teaching video.

5.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Express each of the following as a natural logarithm,

(i) $arcosh\left(3\right)$

[1 mark]

(ii) $arcosh\left(\sqrt{3}\right)$

[1 mark]

(iii) $arcosh\left(3\sqrt{2}\right)$

[2 marks]

Question 2

Find the exact value of $arcosh\left(\frac{13}{5}\right)$ in as simple a form as possible

[3 marks]

Question 3

Solve the equation $cosh\, w = 12$ giving both solutions to 3 decimal places

[3 marks]

Question 4

Further A-Level Examination Question from June 2017, FP3, Q3 (Edexcel)

(a) Using the definitions for $cosh\ x$ in terms of exponentials, show that

$$cosh\ 2x = 2\ cosh^2\ x - 1$$

[3 marks]

(b) Find the exact values of x for which

$$29\ cosh\ x - 3\ cosh\ 2x = 38$$

giving your answers in terms of natural logarithms

[6 marks]

Question 5

Further A-Level Examination Question from June 2008, FP2, Q2 (Edexcel)

Find the values of x for which

$$8\,cosh\,x \,-\, 4\,sinh\,x \,=\, 13$$

giving your answers as natural logarithms.

[6 marks]

Question 6

Further A-Level Examination Question from June 2004, P5. Q1(b) (Edexcel)

Solve,

$$csch\, x - 2\, coth\, x = 2$$

giving your answer in the form $k \ln a$ where k and a are integers

[5 marks]