Lesson 6

Further A-Level Pure Mathematics, Core 2
Maclaurin Series

6.1 Revision

6.2 Useful Information

6.2.1 A Table of Functions and their Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n\,x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x\ \cot x$	Yes
$\sec x$	$\sec x\ \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes
$\text{arccot}\, x$	$-\frac{1}{1+x^2}$	No
$\text{arcsec}\, x$	$\frac{1}{x\sqrt{x^2-1}}$	No
$\text{arccsc}\, x$	$-\frac{1}{x\sqrt{x^2-1}}$	No

6.2.2 The Trigonometric Addition Formula

The Addition Formulae

$$\sin(A \pm B) = \sin A\ \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A\ \cos B \mp \sin A\ \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\ \tan B}$$

6.2.3 Standard Maclaurin Series

Quotable Maclaurin Series Expansions

- $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$ valid for all x
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots$ $-1 < x \leqslant 1$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r + 1)!} + \ldots$ valid for all x
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$ valid for all x
- $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots. + (-1)^r \frac{x^{2r+1}}{2r + 1} + \ldots$ $-1 < x \leqslant 1$

6.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 38

Question 1

Use the addition formula for $\sin(A + B)$ and the series expansions of $\sin x$ and $\cos x$ to show that,

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

[4 marks]

Question 2

Given that $f(x) = \ln \sec x$

(a) show that $f'(x) = \tan x$

[2 marks]

(b) find the values of $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$

[2 marks]

(c) express $\ln \sec x$ as a series in ascending powers of x up to and and including the term in x^4

[2 marks]

(d) show that using the first two non-zero terms of the Maclaurin series for $\ln \sec x$, with $x = \frac{\pi}{4}$, gives a value for $\ln 2$ of $\frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

[2 marks]

Question 3

Further A-Level Examination Question from June 2007, FP2, Q1(c) (MEI) edited

$$f(x) = arccos(2x)$$

(i) Write down $f'(x)$

[2 marks]

(ii) A special case of the binomial theorem is that,

$$(1 + x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ldots \qquad -1 < x \leqslant 1$$

Show how to use this to expand $f'(x)$, and hence find the Maclaurin series for $f(x)$ in ascending powers of x up to and including the term in x^5

[4 marks]

Question 4

Obtain the first three terms of the Maclaurin series for $f(x) = arccot(1 + x)$

[5 marks]

Question 5

Further A-Level Examination Question from January 2009, FP2, Q1(a)(ii) (MEI)

The Maclaurin expansion of $\sec x$ begins

$$1 + ax^2 + bx^4$$

where a and b are constants.

Explain why, for sufficiently small x,

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) \approx 1$$

and hence find the values of a and b

[5 marks]

Question 6

Given that $f(x) = ln\left(x + \sqrt{1 + x^2}\right)$

(i) Show that $\sqrt{1 + x^2}\, f'(x) = 1$

[2 marks]

(ii) Show that $\left(1 + x^2\right) f''(x) + x f'(x) = 0$

[2 marks]

(iii) Show that $\left(1 + x^2\right) f'''(x) + 3x f''(x) + f'(x) = 0$

[2 mark]

(iv) State the values of $f\,(0)$, $f'\,(0)$, $f''\,(0)$ and $f'''\,(0)$

[2 marks]

(v) Give the Maclaurin series for $f\,(x)$ in ascending powers of x up to and including the term in x^3

[2 marks]