Lesson 5

Further A-Level Pure Mathematics, Core 2

Maclaurin Series

5.1 Techniques

When the function for which a Maclaurin expansion is sought includes an exponential, the calculation of $f(0)$, $f'(0)$, $f''(0)$, ... $f^{(r)}(0)$ often has a repetitive component. A technique to speed up such calculations is the subject of this lesson.

To illustrate the method a simple example will be used that could be done by other methods that, in this case, would be more efficient. However, the exercise will look at situations where the "other method" fail, or runs into algebraic complications. A key skill in this topic is an ability to switch between solution methods when one approach runs into technical issues.

5.2 Example

Find a sextic approximation for the function $f(x) = x\, e^{x}$ centred on $x = 0$

Teaching Video : http://www.NumberWonder.co.uk/v9098/5.mp4

[8 marks]

5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 54

Question 1

(i) Given that $f(x) = e^{1-2x}$ show that $f'''(x) = -8f(x)$

[3 marks]

(ii) Deduce a general result for $f^{(n)}(x)$

[2 marks]

(iii) Deduce a general result for $f^{(n)}(0)$

[1 mark]

(iv) Determine a sextic polynomial approximation to $f(x)$ centred on $x = 0$

[3 marks]

(v) Substitute $(1 - 2x)$ into the standard Maclaurin series for the exponential function. That is,

- $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^r}{r!} + \ldots$ valid for all x

There is no need to expand any brackets.

[1 mark]

(vi) If the brackets in part(ii) were now to be expanded, the Maclaurin series would not be obtained in a managable way. The substitution method fails. Briefly explain why.

[2 marks]

(vii) Part (vi) illustrates a problem that can arise with the substitution method of obtaining a Maclaurin series from one of the standard expansions in the examination formula book. In this case, a workaround would be to rewrite the function as $f(x) = e \times e^{-2x}$

Show how this now allows the substitution method to work and obtain the same series as was found in part (iv).

Note that in other such questions a simple workaround may not be possible.

[2 marks]

(viii) By letting $x = 0$ in your part (v) answer, obtain a remarkable result involving the reciprocals of factorials.

[2 marks]

Question 2

Given that, $f(x) = e^{3x} - e^{-3x}$

(i) Show that $f''(x) = 3^2 f(x)$

[2 marks]

(ii) Show that $f^{(4)}(x) = 3^4 f(x)$

[2 marks]

(iii) Hence explain why the Maclaurin series for $f(x)$ will not contain any terms where x is even powered

[2 marks]

(iv) Find the first three non-zero terms of the Maclaurin series for $f(x)$ giving each coefficient in its simplest form.

[3 marks]

(v) Find an expression the the n^{th} non-zero term of the Maclaurin series for $f(x)$

[2 marks]

Question 3

Further A-Level Examination Question from 2018 Mock Core 1 Paper, Q1, (Edexcel)

Given that $f(x) = e^{2x} \cos x$

(a) Show that $f''(x) = p f(x) + q f'(x)$
where p and q are integers to be determined

[5 marks]

(b) Hence find the Maclaurin series for $f(x)$, in ascending powers of x, up to and including the term in x^5, giving each coefficient in its simplest form.

[3 marks]

Question 4

Given that, $f(x) = g(x) + h(x)$ where $g(x) = e^x \sin x$ and $h(x) = e^x \cos x$

obtain expressions for $f'(x)$, $f''(x)$, ... $f^{(5)}(x)$ each in the form $p\,g(x) + q\,h(x)$ where p and q for each expression are to be determined.

Hence find the Maclaurin series for

Hence find the Maclaurin series for $f(x)$, in ascending powers of x, up to and including the terms in x^5 giving each coefficient in its simplest form.

[8 marks]

Question 5

Given that $f(x) = ln(1 + cos\,2x) \qquad 0 \leqslant x < \frac{\pi}{2}$

Show that,

(i) $\quad f'(x) = -2\,tan\,x$

[2 marks]

(ii) $\quad f''''(x) = -\left(f'''(x)\, f'(x) + \left(f''(x)^2 \right) \right)$

[5 marks]

(iii) Find the Maclaurin series expansion of $f(x)$, in ascending powers of x, up to and including the term in x^4

[4 marks]