Lesson 3

3.1 Famous Maclaurin Series

Several Maclaurin series are regarded as quotable and may be used without showing how they are derived. In the examination they are given in the formulae booklet;

Quotable Maclaurin Series Expansions

- $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$ valid for all x
- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots$ $-1 < x \leqslant 1$
- $sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots$ valid for all x
- $cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$ valid for all x
- $arctan\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots. + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots$ $-1 < x \leqslant 1$

In addition the binomial expansions of $f(x) = (1 + x)^n$ where n is fractional or negative and $|x| < 1$ are also the Maclaurin series expansion of $f(x)$. Thus the binomial theorem gives further Maclaurin series.

One worth knowing is the geometric series expansion of $\frac{1}{1 - x}$ which is,

- $(1 - x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$ $-1 < x < 1$

3.2 Convergence

In obtaining polynomial approximations to functions, what is sought is the situation in which the first few terms of the polynomial provide a good model of the function. Then, “throwing away” the infinite number of other, higher order terms, will make a minimal difference to the approximation model for values of x close to $x = 0$.
If the Maclaurin series is convergent for all values of x, such as is the case with the functions *sin x* and *cos x*, for example, then taking more terms yields a polynomial that follows the function over an increasing domain. The series is valid for all x.

However, for some Maclaurin series, convergence is only a feature of the series for a restricted domain. Going beyond these values means no higher order terms can be thrown away because they are no longer insignificant. No approximation of the full series can then be made. This is why some of the series in the “Quotable Maclaurin Series Expansions” table have a restricted domain, thus indicating the values of x for which truncating the series yields a valid approximating polynomial.

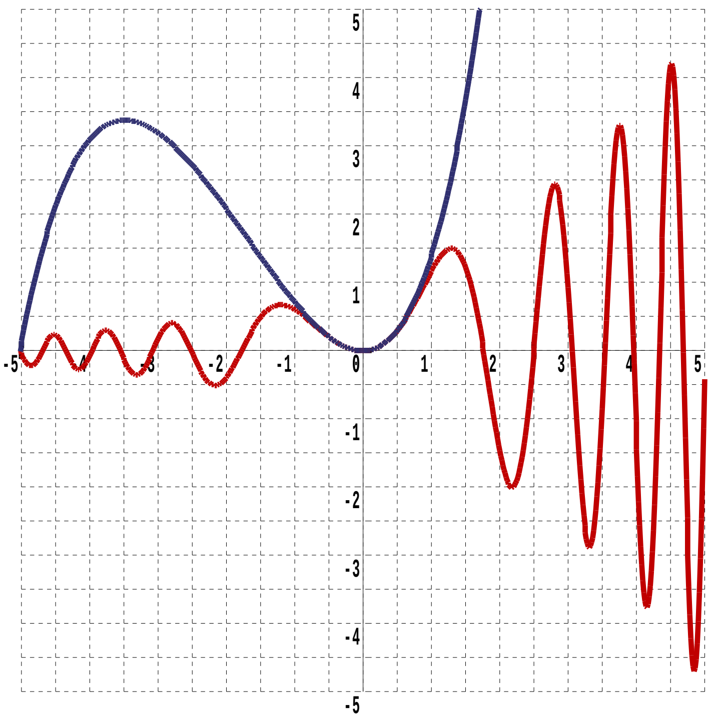
3.3 Substitution

Given an unfamiliar function, and the need to obtain its Maclaurin series, time can often be saved by looking to see if it is sufficiently similar to any of the series in the Quotable Maclaurin Series Expansions table to allow a simple substitution to be made. This can side step having to do the several differentiations that would otherwise be required to obtain an approximating polynomial.

Example

Find the best quintic approximation to $f(x) = e^{\frac{x}{\pi}} sin(x^2)$ centred on $x = 0$

Teaching Video : http://www.NumberWonder.co.uk/v9098/3.mp4



In red is the graph of $y = e^{\frac{x}{\pi}} sin(x^2)$ and in blue the best quintic polynomial approximation on $x = 0$

[6 marks]

3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

(i) By means of a suitable substitution into the Maclaurin series for e^x, find a quintic polynomial approximation to the function $g(x) = \dfrac{1}{e^x}$

(ii) State the values of x for which the expansion is valid

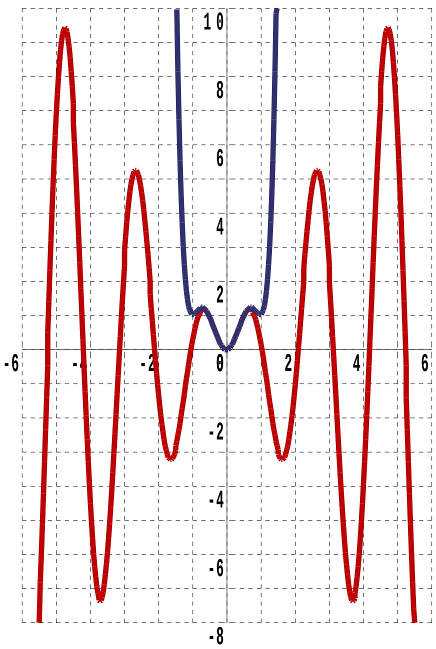
[3 marks]

Question 2

(i) By means of a suitable substitution into the Maclaurin series for $\sin x$, find a sextic (degree 6) polynomial approximation to the function

$$h(x) = 2x \sin(3x)$$

(ii) State the values of x for which the expansion is valid.



In red is the graph of the function $h(x) = 2x \sin(3x)$
In blue is the sextic polynomial approximation on $x = 0$

[3 marks]

Question 3

(i) The Maclaurin series for *arctan x* states that,

- $arctan\, x \;=\; x \;-\; \dfrac{x^3}{3} \;+\; \dfrac{x^5}{5} \;-\; .... \;+\; (-1)^r \;\dfrac{x^{2r+1}}{2r \;+\; 1} \;+\; ... \qquad -1 \;<\; x \;\leqslant\; 1$

By substituting $x = 1$ into both sides of this Maclaurin series deduce an astonishing result that gives a formula for π in terms of a multiple of an alternating series of reciprocal odd numbers.
This result is known as the Leibniz formula for π although it was first discovered long before Leibniz by the Indian mathematician Madhava of Sangamagrama in the 14th century.

[2 marks]

(ii) Determine the value of π given by using the first five terms of the series.
The Leibniz formula for π converges extremely slowly.
To get π accurate to 10 decimal places requires five billion terms !

[2 marks]

Question 4

By means of a suitable substitution into the geometric series,

- $(\,1 \;-\; x\,)^{-1} \;=\; 1 \;+\; x \;+\; x^2 \;+\; ... \;+\; x^r \;+\; ...$ valid for $-\;1 \;<\; x \;<\; 1$

find a octic (degree 8) polynomial approximation to the function $g\,(x) \;=\; \dfrac{1}{1 \;+\; x^2}$

[3 marks]

Question 5

Further A-Level Examination Question from June 2009, FP2, Q1 (a) (MEI)

(i) Use the Maclaurin series for $ln(1 + x)$ and $ln(1 - x)$ to obtain the first three non-zero terms in the Maclaurin series for $ln\left(\frac{1 + x}{1 - x}\right)$

State the range of validity of this series.

[4 marks]

(ii) Find the value of x for which $\frac{1 + x}{1 - x} = 3$

Hence find an approximation to $ln\ 3$, to three decimal places.

[4 marks]

Question 6

(i) Use a standard Maclaurin series, together with a suitable substitution, to find a quartic polynomial approximation for the function,

$$f(x) = (1 - 3x)\ ln(1 + 2x)$$

(ii) State the values of x for which the expansion is valid

[4 marks]

Question 7

(i) By writing $2 + 3x$ as $2\left(1 + \frac{3x}{2}\right)$, use a standard Maclaurin series, together with a suitable substitution, to find a quintic polynomial approximation to, $g(x) = 64x\ ln(2 + 3x)$

(ii) State the values of x for which the expansion is valid

[4 marks]

Question 8

(i) Write down the first five non-zero terms in the series expansion of $e^{-\frac{x^2}{2}}$

[3 marks]

(ii) Using your result from part (a), find an approximate value for

$$\int_{-1}^{1} e^{-\frac{x^2}{2}}\, dx$$

giving your answer to 3 decimal places

[3 marks]

Question 9

Further A-Level Examination Question from January 2010, FP2, Q2 (c) (MEI)

Write down the Maclaurin series for e^t as far as the term in t^2

Hence show that, for t close to zero,

$$\frac{t}{e^t - 1} \approx 1 - \frac{1}{2}t$$

[5 marks]