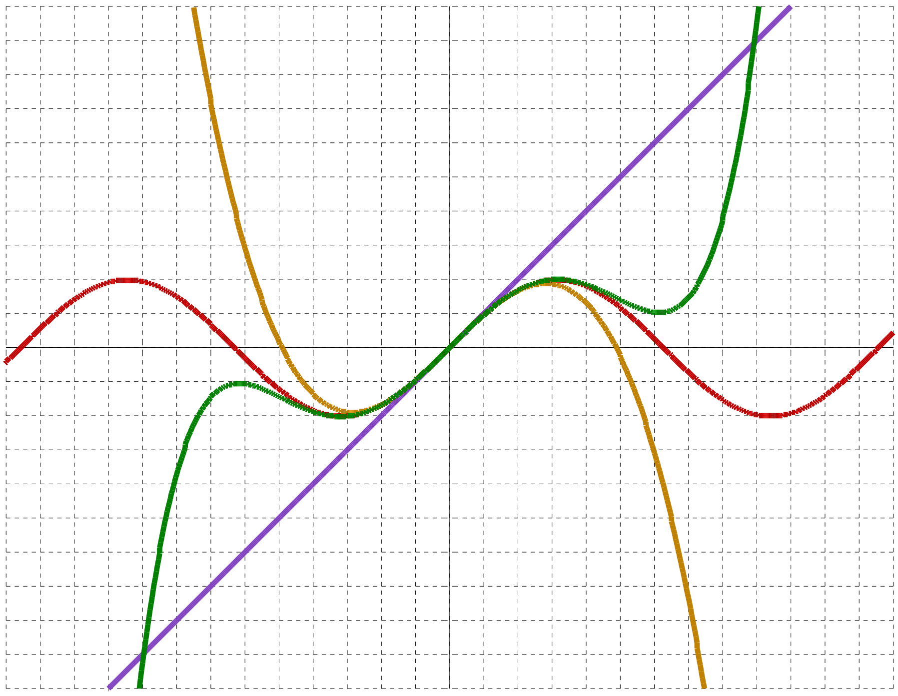
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

MACLAURIN SERIES



~ In search of a polynomial that will model the sine function ~

Purple : $y = x$

Gold : $y = x - \frac{x^3}{3!}$

Green : $y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

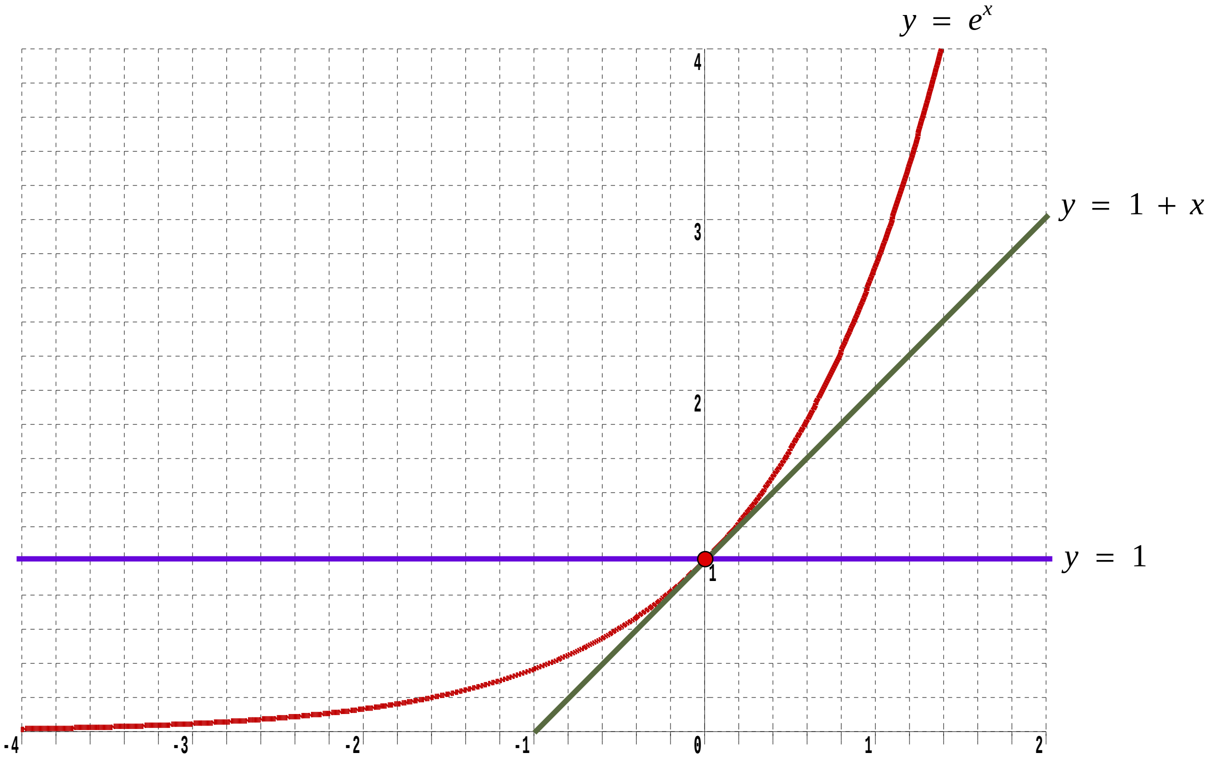
M A C L A U R I N
S E R I E S

Lesson 1

1.1 Approximation

Polynomials are straightforward to differentiate or integrate and have many properties that make them attractive to mathematicians such as, for example, a polynomial of degree n having n roots over the complex numbers.
These desirables provide an incentive to approximate other functions, that are less easy to analyse, with polynomials.

As an example, consider the exponential function and suppose that the interest is in finding the best polynomial approximation of various degrees with the approximations centring on where the function crosses the y-axis.
The best polynomial of degree 0 would simply be $y = 1$
The best polynomial of degree 1 would be $y = 1 + x$



The first approximation, $y = 1$, got the height of where $y = e^x$ crossed the y-axis correct, whereas the second approximation, $y = 1 + x$, also got the gradient of the exponential curve correct at that point.

The third approximation will need to get point, gradient and bend at (0, 1) correct. Intuitively, the best quadratic curve is being sought that will model the exponential and it should be the most successful yet at following the exponential for points that are not too far away from (0, 1).

1.2 The Third Approximation

The exponential function has the remarkable property of being its own derivative.

$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = e^x$$

and so, when $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$ and $\frac{d^2y}{dx^2} = 1$

Let the best quadratic approximation be,

$$y = a_0 + a_1 x + a_2 x^2$$

It is required that $y = 1$ when $x = 0$ and so $a_0 = 1$

Differentiating the best quadratic approximation gives,

$$\frac{dy}{dx} = a_1 + 2 a_2 x$$

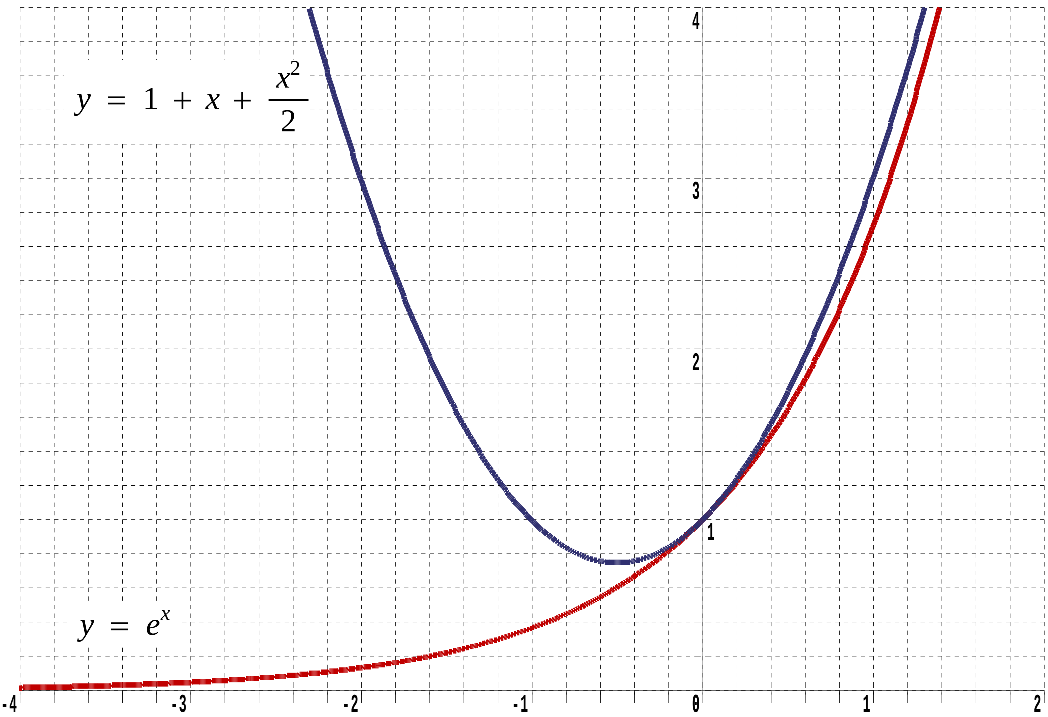
It is required that $\frac{dy}{dx} = 1$ when $x = 0$ and so $a_1 = 1$

Differentiating the best quadratic approximation twice gives.

$$\frac{d^2y}{dx^2} = 2 a_2$$

It is required that $\frac{d^2y}{dx^2} = 1$ when $x = 0$ and so $a_2 = \frac{1}{2}$

The third approximation is thus, $y = 1 + x + \frac{x^2}{2}$



1.3 The Fourth Approximation

The approximations studied so far are,

$$y_0 = 1$$

$$y_1 = 1 + x$$

$$y_2 = 1 + x + \frac{x^2}{2}$$

Notice that each higher order approximation contains those previous to it.
In consequence, for the fourth approximation the working can be shortened;

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

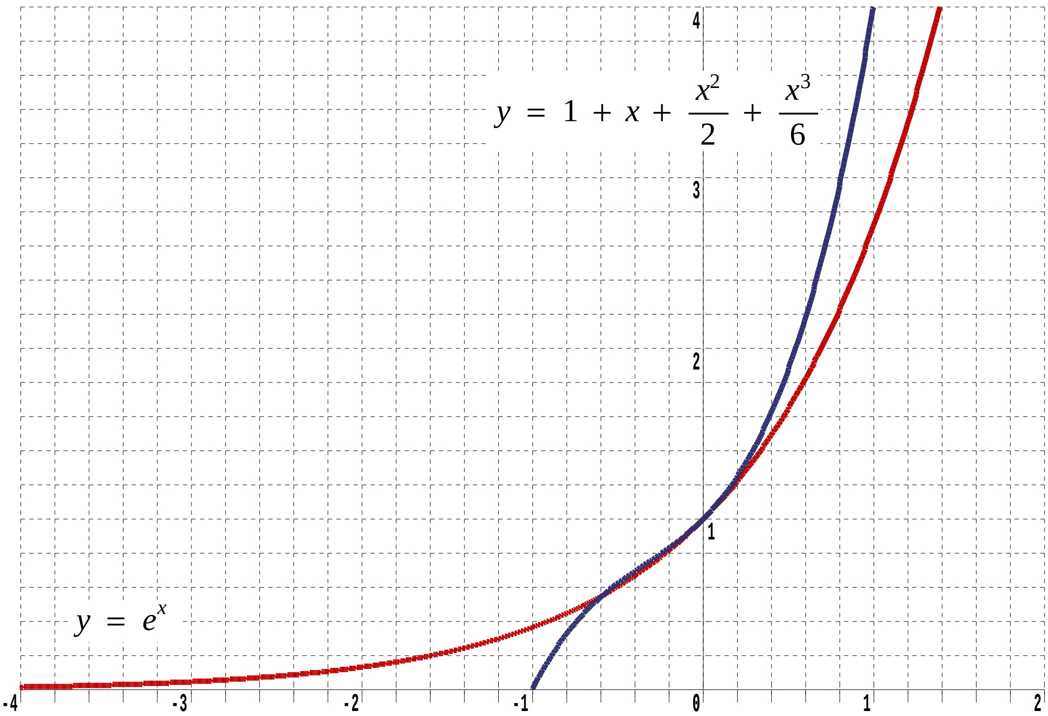
$$\frac{dy}{dx} = \ldots + 3 a_3 x^2$$

$$\frac{d^2y}{dx^2} = \ldots + 6 a_3 x$$

$$\frac{d^3y}{dx^3} = \ldots + 6 a_3$$

It is required that $\dfrac{d^3y}{dx^3} = 1$ when $x = 0$ and so $a_3 = \dfrac{1}{6}$

The fourth approximation is thus, $y = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$



Continuing in this fashion, the exponential function can be approximated to ever more accuracy by taking more and more terms from the infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad x \in \mathbb{R}$$

1.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

A general polynomial of degree four, a quartic, can be expressed in the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Complete the table to show the first, second, third and fourth derivatives of *y*

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$

[4 marks]

Question 2

Complete the table to show the first, second, third and fourth derivatives of the function $y = cos\ x$ and also the values of those derivatives at $x = 0$

$y = cos\ x$	when $x = 0$,	$y = 1$
$\frac{dy}{dx} =$	when $x = 0$,	$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$	when $x = 0$,	$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$	when $x = 0$,	$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$	when $x = 0$,	$\frac{d^4y}{dx^4} =$

[4 marks]

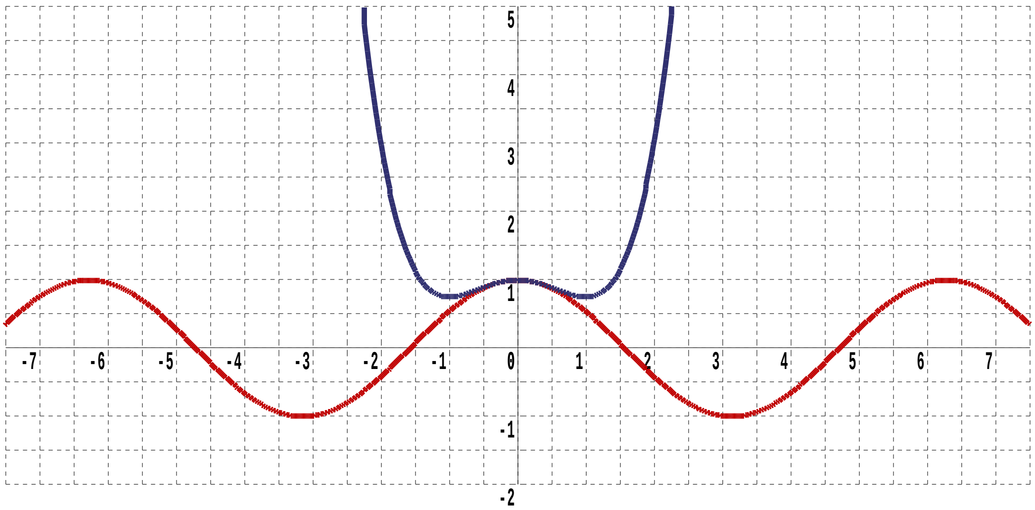
Question 3

By combining the tables of question 1 and 2, work out, in the following order, a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quartic polynomial to represent the cosine function.

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the cosine curve as another (red).

Remember that, as this mathematics is a mixture of calculus and trigonometry, the units of angle must be radians.

Here is an image of what my graph plotter gives;



[6 marks]

Question 4

A general polynomial of degree five, a quintic, can be expressed in the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Complete the table to show the first, second, third, fourth and fifth derivatives of y

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$
$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$

[4 marks]

Question 5

Complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = sin\, x$ and also the values of those derivatives at $x = 0$

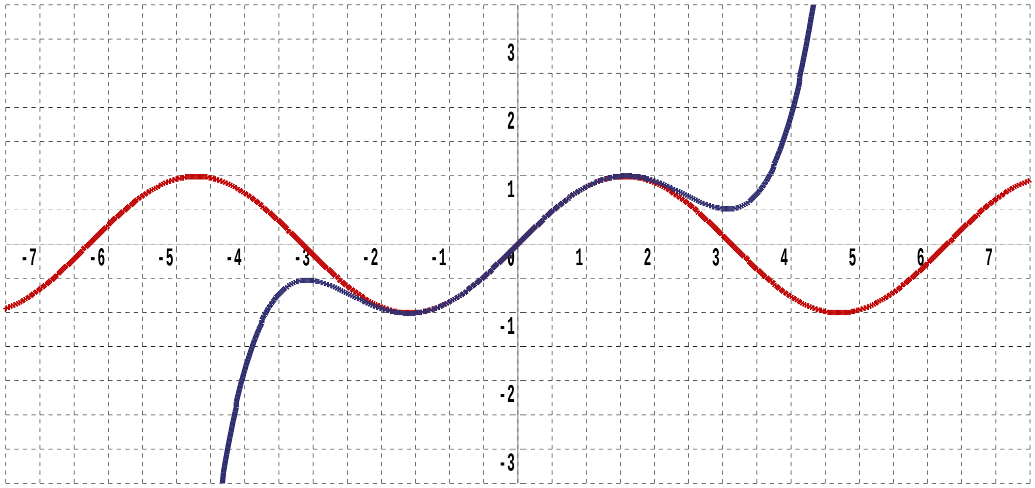
$y = sin\, x$	when $x = 0$,	$y = 0$
$\frac{dy}{dx} =$	when $x = 0$,	$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$	when $x = 0$,	$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$	when $x = 0$,	$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$	when $x = 0$,	$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$	when $x = 0$,	$\frac{d^5y}{dx^5} =$

[5 marks]

Question 6

By combining the tables of questions 4 and 5, work out, in the following order, a_5 then a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quintic polynomial to represent the sine function.

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the sine curve as another (red).



[6 marks]

Question 7

Taking care over the minus signs, and making use of the chain rule, complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = ln(1 - x)$ where $x < 1$ and also the values of those derivatives at $x = 0$

$y = ln(1 - x)$	when $x = 0$,	$y = 0$
$\frac{dy}{dx} =$	when $x = 0$,	$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$	when $x = 0$,	$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$	when $x = 0$,	$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$	when $x = 0$,	$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$	when $x = 0$,	$\frac{d^5y}{dx^5} =$

[5 marks]

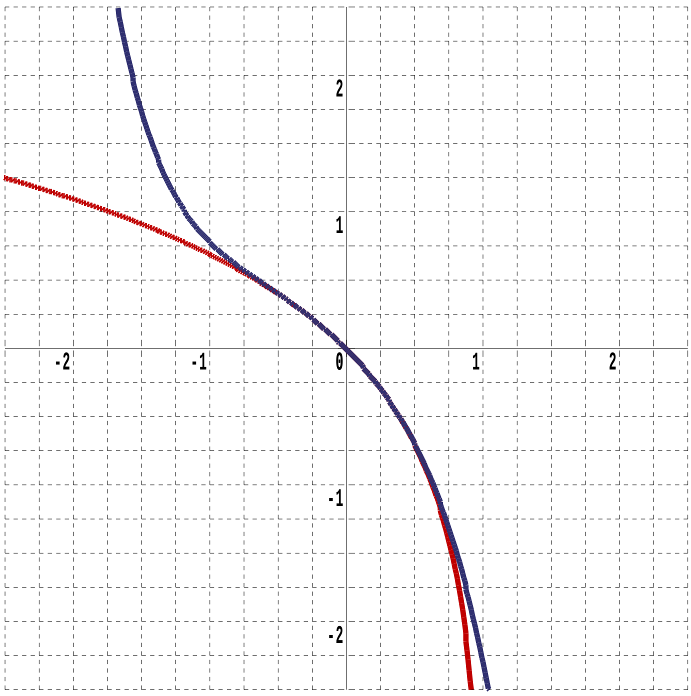
Question 8

$y = ln(1 - x)$ is to be approximated by

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

By combining the tables of questions 4 and 7 work out, in the following order, a_5 then a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quintic polynomial to represent the function $y = ln(1 - x)$

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the curve $y = (1 - x)$ as another (red)



[6 marks]