7.1 Describing a Three Dimensional Plane

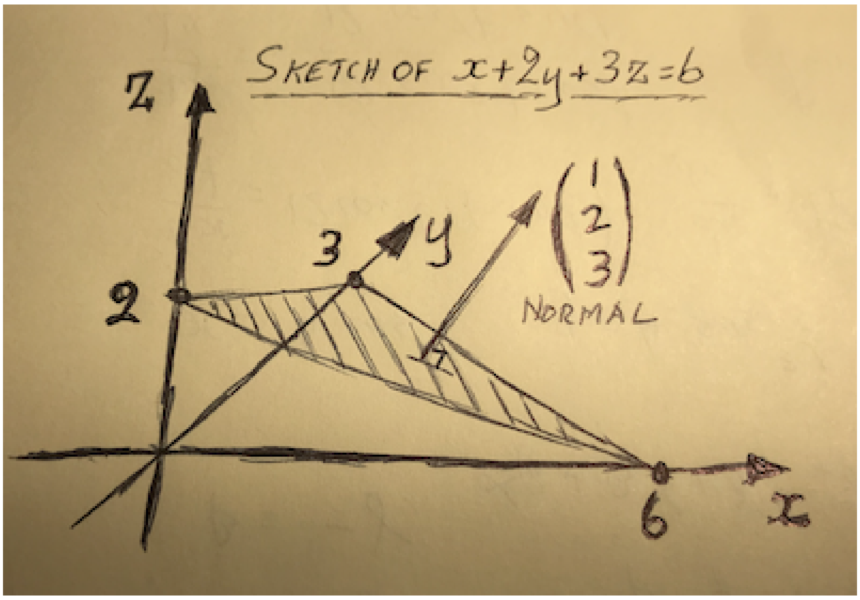
In three dimensional Euclidean space, a plane has an equation of the form,

$$ax + by + cz = d \text{ where } a, b, c \text{ and } d \text{ are constants}$$

For example the equation $x + 2y + 3z = 6$ describes a plane.
It can be sketched by realising that,

- it will cross the x-axis when y and z are zero; (6, 0, 0)
- it will cross the y-axis when x and z are zero; (0, 3, 0)
- it will cross the z-axis when x and y are zero; (0, 0, 2)

which leads to,



A plane can be considered to "look" in the direction given by its normal, which gives us a means of detecting parallel planes; they will have normals that look in the same direction (or the 180° opposite direction).

The Three-Dimensional Plane

A Cartesian equation of a plane in three dimensions can be written in the form

$$ax + by + cz = d \text{ where } a, b, c \text{ and } d \text{ are constants.}$$

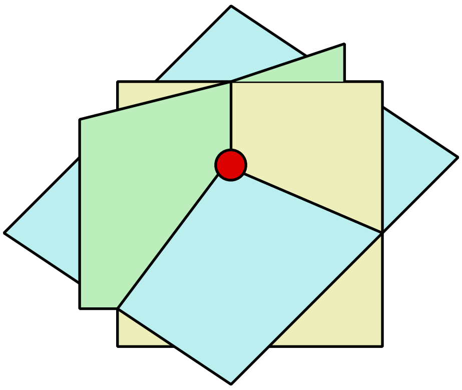
The vector $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal to the plane and, intuitively, gives the in which the plane is "looking".
Parallel planes look either in the same direction, or 180° opposite directions.
The *unit normal* is a vector in the normal's direction of length 1 unit.

7.2 Point Intersection of Three Planes

Suppose that we have the following three planes,

$$\begin{aligned} 2x - 3y + z &= -6 \\ x + 2y - 4z &= 12 \\ 3x + y + 2z &= 7 \end{aligned}$$

Typically, three planes chosen at random will intersect in a single common point and such planes are said to be “in general position” as illustrated below.



The point of intersection is most easily found by setting up and solving a matrix system of equations. Use a calculator if all of the entries in the matrix are numbers.

For the above three planes we'd proceed as follows;

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -4 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{53}\begin{pmatrix} 8 & 7 & 10 \\ -14 & 1 & 9 \\ -5 & -11 & 7 \end{pmatrix}\begin{pmatrix} -6 \\ 12 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

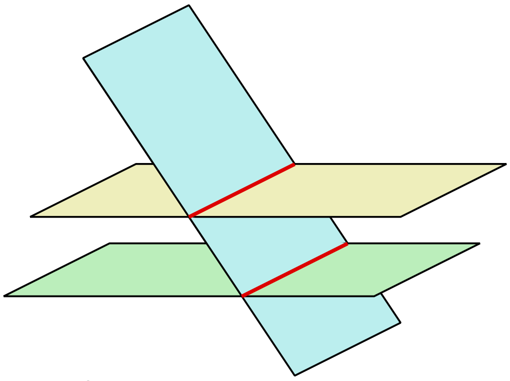
So the three planes intersect at the point $(2,\ 3,\ -1)$

Notice that the intermediate line of working, showing the inverse matrix, should be given. However, most of the calculation is all in the calculator with nothing written down.

7.3 Special Case Three Plane Intersections

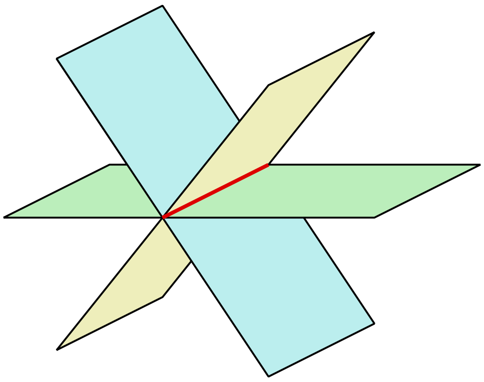
There are only two types of configuration with two planes; either the planes are parallel and they do not intersect, or else they intersect in a common line.

With three planes, there are a total of five situations that can arise. The first of these, where the the planes are in general position, has been dealt with but there are four special cases to consider, as illustrated below.



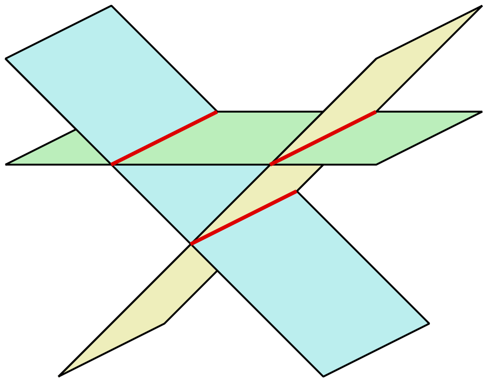
Two of the planes (only) are parallel

All three planes are parallel



The planes form a sheaf

They share a common line



The planes form a prism

They meet in pairs in parallel lines

With all of these different possibilities to worry about, we need a general strategy that will work through then in a brisk and efficient fashion. This is in three steps and is presented next.

Step 1 : Test for Planes in General Position

Unless you have reason to believe one of the other configurations is in play, test first to see if the three planes are in general position with a single point of intersection through which all three planes pass. Do this by expressing the equations of the three planes as one 3×3 matrix **S** and consider it to be a system of equations with a solution that is the point of intersection (x, y, z). If the determinant of **S** is not zero then **this is good news**; there is a unique solution, a single point, and the planes are in general position. Solve the system of equations to find this point.

If the determinant is zero then one of the other four situations applies and it is necessary to proceed to step 2.

Step 2 : Test for Parallel Planes

Having found that the determinant of the system of equations is zero, the next test is to find out if two or three of the planes are parallel. Do this by looking at the directions of the normal vectors. The number of normal vectors in the same direction (or a 180° opposite direction match) gives the number of parallel planes. The 'raw' normal vectors will likely be of different lengths (unless you process them to make them all of unit length) so take care not miss any that are in the same direction. Unfortunately, the A-Level course does not teach you how to do this using matrices. Beyond A-Level, you calculate something called the rank of a matrix and you may like to explore Wolfram Alpha's **MatrixRank** command to help determine the number of normal vectors that are in the same direction.

If no parallel planes are detected one of the remaining two situations applies and it is necessary to proceed to step 3

Step 3 : Testing for a Sheaf or Prism

Having worked through steps 1 and 2 without identifying the configuration, only two possibilities remain. We can test for which in one go. Combine one of the three equations separately with each the other two. For example, say the first is added to the second and, separately, the first is added to the third. If a consistency is obtained then a sheaf has been detected. If an inconsistency is obtained then a prism has been detected. Alas, once again the A-Level course does not develop the topic of matrices far enough to do this by “Gaussian elimination” so you need to do it using algebraic wits.

7.4 An Example

Consider the three planes with equations

$$
\begin{aligned}
x + 2y + 3z &= 13 \\
-x - 3y + 2z &= 2 \\
-x - 4y + 7z &= 9
\end{aligned}
$$

(i) Set up a matrix system of equations and then show that the 3×3 matrix involved has a determinant of zero. State what this tells you about the configuration of the planes.

[4 marks]

(ii) Write down the normals for each plane.
Explain how these tell you none of the planes are parallel.

[4 marks]

(iii) Add together the upper and middle equations.
Separately, add together the upper and lower equations.
Show that an inconsistency results.
In which configuration are the three planes ?

[4 marks]

7.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 45

Question 1

By setting up a matrix system of equations, find the point of intersection of the following three planes using the matrix facility on your calculator.

$$\begin{aligned} x + 4y - 5z &= 8 \\ 2x + 3y + z &= -1 \\ 3x - y + 2z &= 3 \end{aligned}$$

[4 marks]

Question 2

Three planes have the equations,

$$\begin{aligned} x - 2y + z &= 5 \\ 3x - 6y + 3z &= 12 \\ x + 2y - z &= -2 \end{aligned}$$

Kevin has worked out the the associated matrix has a determinant of zero.
What is the configuration of these planes ?
Give a reason for your answer.

[4 marks]

Question 3

FM A-Level Examination Question from October 2020, Paper Core Pure, Q15 (MEI)

(a) Show that the three planes with equations,

$$\begin{aligned} x + \lambda y + 3z &= -12 \\ 2x + y + 5z &= -11 \\ x - 2y + 2z &= -9 \end{aligned}$$

where λ is a constant, meet at a unique point except for one value of λ which is to be determined.

[3 marks]

(b) In the case $\lambda = -2$, use matrices to find the point of intersection of the planes, showing your method clearly.

[3 marks]

Question 4

Determine the configuration of the following three planes,

$$\begin{aligned} x + 4y - 5z &= 8 \\ 2x + 3y + z &= -1 \\ 3x + 7y - 4z &= 5 \end{aligned}$$

[4 marks]

Question 5

FM A-Level Examination Question from October 2021, Paper Core Pure, Q15 (MEI)

The equations of three planes are,

$$-4x + py + 7z = 4$$

$$x - 2y + 5z = q$$

$$2x + 3y + z = 2$$

Given that the planes form a sheaf, determine the values of p and q

[6 marks]

Question 6

FM A-Level Examination Question from October 2020, Paper Core Pure, Q9 (MEI)

Three planes have equations,

$$kx + y - 2z = 0 \qquad \text{where } k \text{ is a constant}$$

$$2x + 3y - 6z = -5$$

$$3x - 2y + 5z = 1$$

Investigate the arrangement of the planes for each of the following cases.

If in either case the planes meet at a unique point, find the coordinates of that point.

(a) $k = -1$

[3 marks]

(b) $k = \frac{2}{3}$

[4 marks]

Question 7

FM A-Level Examination Question from June 2019, Paper Core Pure, Q14 (MEI)

Three planes have equations,

$$\begin{aligned} -x + ay \quad &= 2 \\ 2x + 3y + z &= -3 \\ x + by + z &= c \end{aligned}$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a

[4 marks]

(ii) find the value of c for which the planes form a sheaf.

[3 marks]

(b) In the case where $b = a$ and $c = 1$, find the coordinates of the point of intersection of the planes in terms of a.

[6 marks]

Question 8

FM A-Level Examination Question from June 2021, Paper 1, Q12 (AQA)

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 \\ 4 & -2 & p \\ 8 & 5 & -11 \end{pmatrix}$, where p is a constant.

(a) Given that $\mathbf{A}$ is a non-singular matrix, find $\mathbf{A}^{-1}$ in terms of p.

State any restrictions on the value of p.

[6 marks]

(b) The equations below represent three planes,

$$\begin{aligned} x + 5y + 3z &= 5 \\ 4x - 2y + pz &= 24 \\ 8x + 5y - 11z &= -30 \end{aligned}$$

(i) Find, in terms of p, the coordinates of the point of intersection of the three planes.

[4 marks]

(ii) In the case where $p = 2$, show that the planes are mutually perpendicular.

[4 marks]