Lesson 8

8.1 Extension Material

In order to solve a general cubic equation, Cardano first reduced it to depressed cubic form,

$$t^3 + pt + q = 0$$

This provides a motivation to look at the algebra of the roots of this equation which has the special property of $\alpha + \beta + \gamma = 0$.

The general result that,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

becomes,

$$\alpha^2 + \beta^2 + \gamma^2 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

and another general result that,

$$\alpha^3 + \beta^3 + \gamma^3$$
$$= (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

becomes

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$

In this extension lesson a further couple of formulae are derived and then a question given in which applying them may be practiced.

A Cubic's Sum of Squares of Product Pairs of Roots

If α, β and γ are roots of $a x^3 + b x^2 + cx + d = 0$ $\quad a, b, c, d \in \mathbb{C}$

then, $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

For a depressed cubic,

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

Proof

$$P^2 + Q^2 + R^2 = (P + Q + R)^2 - 2(PQ + QR + RP)$$

$$(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta)$$
$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \quad \square$$

For a depressed cubic, $\alpha + \beta + \gamma = 0$

$$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 \quad \square$$

A Cubic's Sum of Cubes of Product Pairs of Roots

If α, β and γ are roots of $a x^3 + b x^2 + cx + d = 0$ $\quad a, b, c, d \in \mathbb{C}$

then,

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$$
$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2$$

For a depressed cubic,

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^3$$

Proof

$$P^3 + Q^3 + R^3 = (P + Q + R)^3 - 3(P + Q + R)(PQ + QR + RP) + 3PQR$$

$$(\alpha\beta)^3 + (\beta\gamma)^3 + (\gamma\alpha)^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3$$
$$- 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta) + 3(\alpha\beta)(\beta\gamma)(\gamma\alpha)$$

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3$$
$$- 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2 \qquad \square$$

For a depressed cubic, $\alpha + \beta + \gamma = 0$

$$\therefore \ \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^2 \qquad \square$$

A Cubic's Sum of Quartics of Roots

If α, β and γ are roots of $ax^3 + bx^2 + cx + d = 0$ $\qquad a,\ b,\ c,\ d \in \mathbb{C}$

then, $\quad \alpha^4 + \beta^4 + \gamma^4$

$$= \left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2$$
$$- 2\left((\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \right)$$

For a depressed cubic,

$$\alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

Proof

$$P^2 + Q^2 + R^2 = (P + Q + R)^2 - 2(PQ + QR + RP)$$
$$(\alpha^2)^2 + (\beta^2)^2 + (\gamma^2)^2 = (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)$$

$$\alpha^4 + \beta^4 + \gamma^4$$
$$= \left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2 - 2\left(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\right)$$
$$= \left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2$$
$$- 2\left((\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \right) \qquad \square$$

For a depressed cubic, $\alpha + \beta + \gamma = 0$

$$\therefore\ \alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2 \qquad \square$$

8.2 Exercise

Question 1

If α, β and γ are the roots of $x^3 + x + 1 = 0$, then, without solving the equation, find the equation whose roots are $(\alpha - \beta)^2$, $(\beta - \gamma)^2$ and $(\gamma - \alpha)^2$

[8 marks]