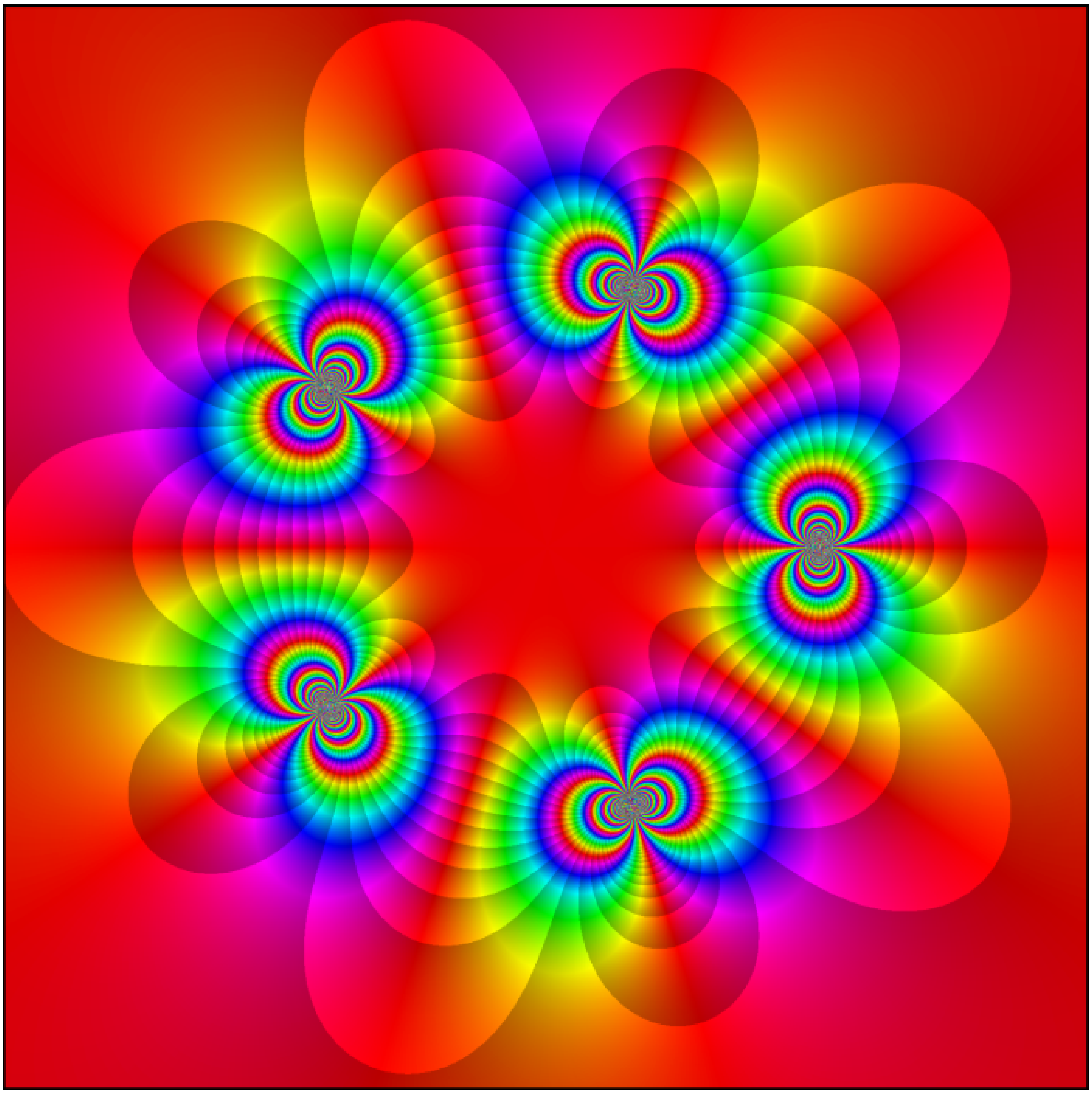
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

ROOTS ~ OF ~ POLYNOMIALS



The five roots in the complex plane of the polynomial equation $z^5 = 1$

ROOTS
OF
POLYNOMIALS

Lesson 1

Further A-Level Pure Mathematics
Roots of Polynomials : Core 1

1.1 Quadratic and Roots

The simple quadratic equation $ax^2 + bx + c = 0$ is a surprisingly rich source of mathematical ideas. It was the original motivation to develop the technique of completing the square, and a doorway into an understanding of complex numbers. Through iterating the simple quadratic $z^2 = c$ the world of Fractal Geometry was discovered in the 1980s. This topic *Roots of Polynomials* also starts by looking at the quadratic equation from a new perspective.

1.2 The Sum Of The Roots

The Sum Of The Roots

If α and β are the roots of the equation $ax^2 + bx + c = 0 \qquad a,\ b,\ c \in \mathbb{C}$

then, $$\alpha + \beta = -\frac{b}{a}$$

Proof

Without loss of generality, let $$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and $$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

then

$$\begin{aligned}
\alpha + \beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
&= -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= -\frac{b}{2a} - \frac{b}{2a} \\
&= \frac{-2b}{2a} \\
&= -\frac{b}{a} \qquad \square
\end{aligned}$$

1.3 The Product Of The Roots

The Product Of The Roots

If α and β are the roots of the equation $a\,x^2 + bx + c = 0 \qquad a,\ b,\ c \in \mathbb{C}$

then, $$\alpha\beta = \frac{c}{a}$$

Proof

Without loss of generality, let $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

then

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4\,a^2} \qquad \text{Difference of two squares}$$

$$= \frac{b^2 - \left(b^2 - 4ac\right)}{4\,a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4\,a^2}$$

$$= \frac{4ac}{4\,a^2}$$

$$= \frac{c}{a} \qquad \square$$

1.4 Sum, Product Example

The roots of the quadratic $8\,x^2 + 2x - 15 = 0$ are α and β.
Without solving the equation, find the values of

(i) $\alpha + \beta$ **(ii)** $\alpha\beta$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(iv)** $\alpha^2 + \beta^2$

Teaching Video : http://www.NumberWonder.co.uk/v9093/1.mp4

1.5 After Watching the Teaching Video

(a) Having watched the teaching video complete the following,

(i) In general $a x^2 + bx + c = 0$ divided throughout by a gives,

☞

[1 mark]

(ii) This is is useful because,

☞

[1 mark]

(iii) For the particular example, $8 x^2 + 2x - 15 = 0$ is divided by 8 to get,

☞

[1 mark]

(b) Without solving the equation, find the values of

(i) $\alpha + \beta$

☞

(ii) $\alpha\beta$

☞

[1, 1 mark]

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

☞

(iv) $\alpha^2 + \beta^2$

☞

[2, 2 marks]

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

The roots of the quadratic equation $3x^2 + 7x - 2 = 0$ are α and β.

Without solving the equation, find the values of

(i) $\alpha + \beta$ **(ii)** $\alpha\beta$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(iv)** $\alpha^2 + \beta^2$

[1, 1, 2, 2 marks]

Question 2

The roots of the quadratic equation $4x^2 - 3x + 1 = 0$ are α and β.

Without solving the equation, find the values of,

(i) $\alpha + \beta$ **(ii)** $\alpha\beta$

(iii) $\alpha^2 + \beta^2$ **(iv)** $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

[1, 1, 2, 2 marks]

Question 3

(i) Prove that,

$$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \alpha^3 + \beta^3$$

[2 marks]

(ii) The roots of the quadratic equation $5x^2 + 6x + 2 = 0$ are α and β.

Without solving the equation, find the exact value of $\alpha^3 + \beta^3$

[4 marks]

Question 4

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{1}{2}$ and $\beta = -\frac{2}{3}$

Find integer values for *a*, *b* and *c*

[3 marks]

Question 5

The complex roots of a quadratic equation $a x^2 + bx + c = 0$ are

$$\alpha = \frac{1 + 2\mathrm{i}}{3} \quad \text{and} \quad \beta = \frac{1 - 2\mathrm{i}}{3}$$

Find integer values for a, b and c

[4 marks]

Question 6

The roots of the equation $6 x^2 + 36x + k = 0$ are reciprocals of each other.

Find the value of the constant, k

[4 marks]

Question 7

Gerolamo Cardano (1501-1576) is credited with the first formula for solving cubic equations.

Depressed Cubic Formation Rule

Faced with a general cubic,

$$a x^3 + b x^2 + cx + d = 0 \qquad a,\ b,\ c,\ d \in \mathbb{C}$$

initiate a change of variable by replacing x with $t - \dfrac{b}{3a}$

This will always result in what is termed a depressed cubic, one of the form,

$$t^3 + pt + q = 0 \qquad p,\ q \in \mathbb{C}$$

(i) Make the appropriate change of variable for the cubic,

$$x^3 - 3x^2 + 12x + 16 = 0$$

and show that the resulting depressed cubic is,

$$t^3 + 9t + 26 = 0$$

[4 marks]

Root of a Cubic

Given a depressed cubic of the form

$$t^3 + pt + q = 0 \qquad p,\ q \in \mathbb{C}$$

where p and q are not both zero, and $4p^3 + 27q^2 \neq 0$, calculate,

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A root of the cubic is then given by,

$$\alpha = C - \frac{p}{3C}$$

(ii) Determine a root of the depressed cubic,

$$t^3 + 9t + 26 = 0$$

[3 marks]

(iii) Using polynomial division, find all three roots, two of which are a complex conjugate pair, of the depressed cubic,

$$t^3 + 9t + 26 = 0$$

[2 marks]

(iv) List the three roots of the original cubic equation,

$$x^3 - 3x^2 + 12x + 16 = 0$$

[2 marks]