### 4.1 The Hole In The Sequence

Mathematician's like integer sequences, and they absolutely love it when there is a mysterious hole in a sequence; a term that should be there but isn't !

Here is a Calculus integer sequence of derivatives;

| Function $f(x)$ | Derivative $f'(x)$ |
|---|---|
| ... | ... |
| $\frac{x^3}{3}$ | $x^2$ |
| $\frac{x^2}{2}$ | $x^1$ |
| $\frac{x^1}{1}$ | $x^0$ |
| The Hole | $x^{-1}$ |
| $\frac{x^{-1}}{-1}$ | $x^{-2}$ |
| $\frac{x^{-2}}{-2}$ | $x^{-3}$ |
| ... | ... |

The reason for the hole is that the function sequence is hitting a division by zero;

$$\ldots, \frac{x^3}{3}, \frac{x^2}{2}, \frac{x^1}{1}, \frac{x^0}{0}, \frac{x^{-1}}{-1}, \frac{x^{-2}}{-2}, \ldots$$

Fortunately, from our work on differentiation†, it is known what is in the hole.

In other words, we know what function differentiates to $\frac{1}{x}$; it's $ln(x)$.

Consequently, the statement of The Chain Rule Backwards can be extended to include this special case;

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**The Chain Rule Backwards**

$$\int f'(x)\ [f(x)]^n\ dx = \frac{[f(x)]^{n+1}}{(n+1)} + c \qquad n \neq -1$$

$$\int f'(x)\ [f(x)]^{-1}\ dx = ln|f(x)| + c \qquad \text{i.e. with } n = -1$$

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† Differentiation III, Lesson 7

**Example**

Determine: $\int \frac{36\,x^2}{4\,x^3 - 9}\,dx$

Teaching Video: http://www.NumberWonder.co.uk/v9045/4.mp4

**[ 3 marks ]**

**4.2 Exercise**

*Any solution based entirely on graphical or numerical methods is not acceptable*

Marks Available : 30

**Question 1**

Determine: $\int \frac{40\,x^3}{1 + 2\,x^4}\,dx$

**[ 3 marks ]**

**Question 2**

Determine: $\int \frac{35\,x^4}{3 - x^5}\,dx$

**[ 3 marks ]**

**Question 3**

Determine: $\int \frac{x^3 + 1}{x^4 + 4x}\,dx$

**[ 4 marks ]**

**Question 4**

Determine: $\int \frac{8\,x^3}{(x^2 + 1)(x^2 - 1)}\,dx$

**[ 4 marks ]**

**Question 5**

**( i )** Explain why finding $\int \frac{x+3}{x^2+x}\,dx$ can not be done by a straight forward application of The Chain Rule Backwards.

**[ 2 marks ]**

**( ii )** Prove that $\frac{x+3}{x^2+x} = \frac{3}{x} - \frac{2}{x+1}$

Begin your proof "RHS ="

**[ 2 marks ]**

**( iii )** Use the part (ii) result to show $\int \frac{x+3}{x^2+x}\,dx = ln\left|\frac{x^3}{(x+1)^2}\right| + c$

**[ 4 marks ]**

**Question 6**

Show that,

$$\int_1^3 \frac{6x^3 + 5x}{3x^4 + 5x^2 + 1}\,dx = ln\left(\frac{17}{3}\right)$$

**[ 8 marks ]**