

**4.1 The Hole In The Sequence**

Mathematicians like integer sequences, and they absolutely love it when there is a mysterious hole in a sequence; a term that should be there but isn't !

Here is a Calculus integer sequence of derivatives;

Function $f(x)$	Derivative $f'(x)$
...	...
$\frac{x^3}{3}$	$x^2$
$\frac{x^2}{2}$	$x^1$
$\frac{x^1}{1}$	$x^0$
The Hole	$x^{-1}$
$\frac{x^{-1}}{-1}$	$x^{-2}$
$\frac{x^{-2}}{-2}$	$x^{-3}$
...	...

The reason for the hole is that the function sequence is hitting a division by zero;

$$\dots, \frac{x^3}{3}, \frac{x^2}{2}, \frac{x^1}{1}, \frac{x^0}{0}, \frac{x^{-1}}{-1}, \frac{x^{-2}}{-2}, \dots$$

Fortunately, from our work on differentiation<sup>†</sup>, it is known what is in the hole.

In other words, we know what function differentiates to  $\frac{1}{x}$ ; it's  $\ln(x)$ .

Consequently, the statement of The Chain Rule Backwards can be extended to include this special case;

**The Chain Rule Backwards**

$$\int f'(x) [f(x)]^n dx = \frac{[f(x)]^{n+1}}{(n+1)} + c \quad n \neq -1$$

$$\int f'(x) [f(x)]^{-1} dx = \ln|f(x)| + c \quad \text{i.e. with } n = -1$$

<sup>†</sup> Differentiation III, Lesson 7

**Example**

Determine:  $\int \frac{36x^2}{4x^3 - 9} dx$

Teaching Video: <http://www.NumberWonder.co.uk/v9045/4.mp4>



[ 3 marks ]

**4.2 Exercise**

*Any solution based entirely on graphical  
or numerical methods is not acceptable*

Marks Available : 30

**Question 1**

Determine:  $\int \frac{40x^3}{1 + 2x^4} dx$

[ 3 marks ]

**Question 2**

Determine:  $\int \frac{35x^4}{3-x^5} dx$

[ 3 marks ]

**Question 3**

Determine:  $\int \frac{x^3 + 1}{x^4 + 4x} dx$

[ 4 marks ]

**Question 4**

Determine:  $\int \frac{8x^3}{(x^2 + 1)(x^2 - 1)} dx$

[ 4 marks ]

**Question 5**

- ( i ) Explain why finding  $\int \frac{x + 3}{x^2 + x} dx$  can not be done by a straight forward application of The Chain Rule Backwards.

[ 2 marks ]

- ( ii ) Prove that  $\frac{x + 3}{x^2 + x} = \frac{3}{x} - \frac{2}{x + 1}$

Begin your proof “RHS =”

[ 2 marks ]

- ( iii ) Use the part (ii) result to show  $\int \frac{x + 3}{x^2 + x} dx = \ln \left| \frac{x^3}{(x + 1)^2} \right| + c$

[ 4 marks ]

**Question 6**

Show that,

$$\int_1^3 \frac{6x^3 + 5x}{3x^4 + 5x^2 + 1} dx = \ln\left(\frac{17}{3}\right)$$

[ 8 marks ]

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In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”

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Teachers may obtain detailed worked solutions to the exercises by email from [mhh@shrewsbury.org.uk](mailto:mhh@shrewsbury.org.uk)