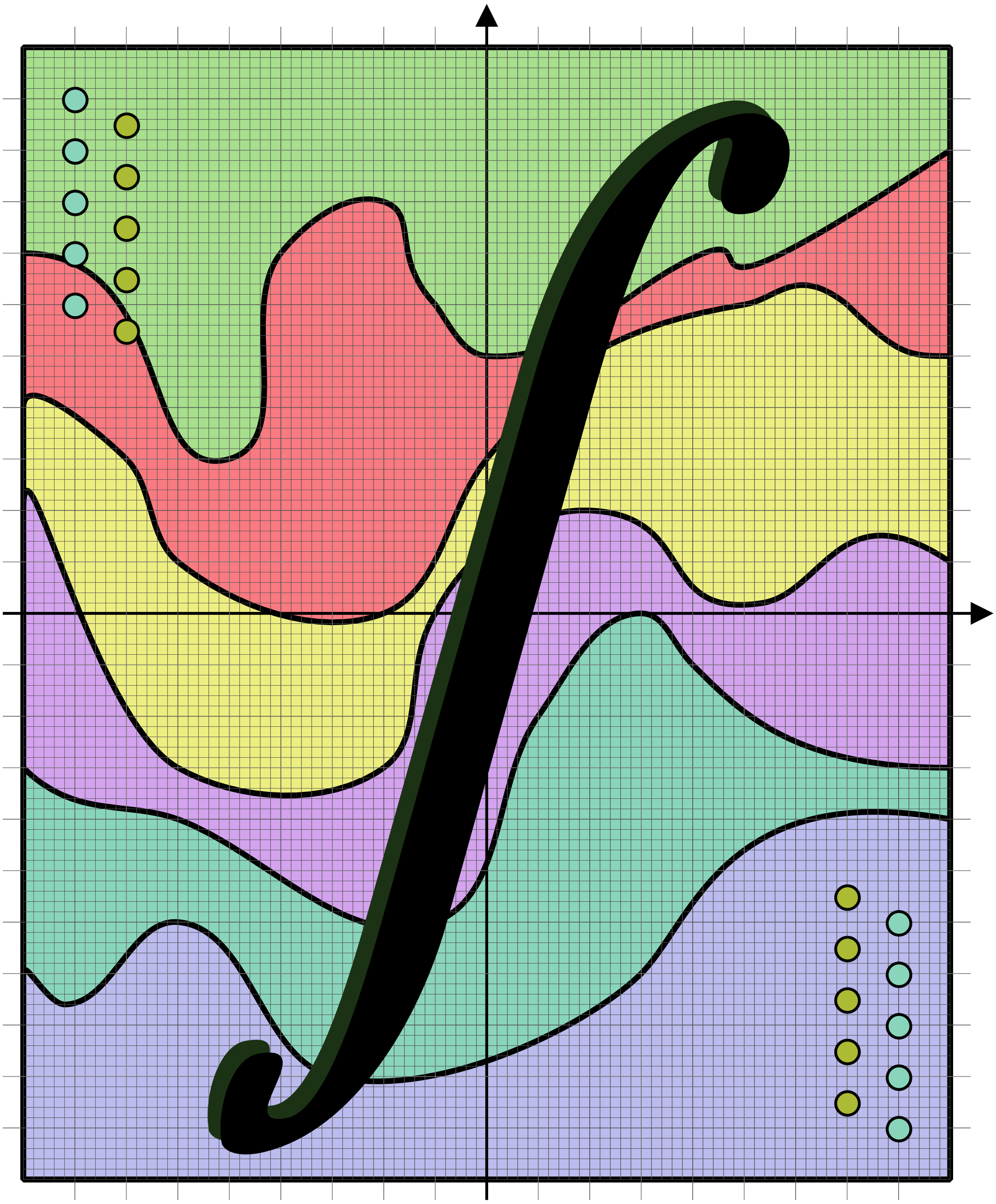
A-Level Pure Mathematics
Year 2

INTEGRATION II



"Areas Between Curves"

1.1 Calculus Compounded

Initial studies of Calculus focus upon the Differentiation and Integration of polynomials using The Power Rule.

The Calculus Power Rule

(i) If $y = x^n$ then $\dfrac{dy}{dx} = n\,x^{n-1}$

(ii) If $y = x^n$ then $\displaystyle\int y \;dx = \frac{x^{n+1}}{n+1} + c$

where n is a constant, and c is “the constant of integration”

In the Year 2 A-Level course, it was discovered that several other elementary functions, not just the polynomials, where ammenable to being differentiated. For example, exponential, logarithmic and trigonometric functions all have derivatives. The chain, product and quotient rules widened further the variety of expressions that could be differentiated.

Here is a reminder of four key results that were obtained, where a is a constant;

Exponential

- If $y = e^{ax}$ then $\dfrac{dy}{dx} = a\,e^{ax}$

Natural Logarithm

- If $y = ln|ax|$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$

Sine

- If $y = sin(ax)$ then $\dfrac{dy}{dx} = a\,cos(ax)$

Cosine

- If $y = cos(ax)$ then $\dfrac{dy}{dx} = -\,a\,sin(ax)$

Our ability to differentiate has become much further advanced than our ability to integrate. The time has come to rebalance this state of affairs !

The key to progressing our knowledge of integration is a result already known;

The Fundamental Theorem of Calculus

The process of integration (finding areas under curves)
is the inverse of
the process of differentiation (finding gradients of curves)

In consequence, each of the four differentiation statements just made can be rephrased as an integration statement simply by “thinking backwards”. Again *a* is a constant and, in addition, as with the polynomials, there will be “a constant of integration”, usually denoted *c*

Exponential

- If $y = e^{ax}$ then $\int y\,dx = \frac{1}{a}e^{ax} + c$

Natural Logarithm

- If $y = \frac{1}{x}$ then $\int y\,dx = ln\,|x| + c$

Sine

- If $y = sin\,(ax)$ then $\int y\,dx = -\frac{1}{a}cos\,(ax) + c$

Cosine

- *If* $y = cos\,(ax)$ then $\int y\,dx = \frac{1}{a}\,sin\,(ax) + c$

Note : **1)** RADIANS must be used whenever trigonometry and calculus mix.
2) With the natural logarithm function a step was missed out

More precisely • If $y = \frac{1}{x}$ then $\int y\,dx = ln|ax| + k$

$$= ln|x| + ln|a| + k$$

$$= ln|x| + c$$

because one unknown number, $ln|a|$, plus a second unknown number, *k*, add together to give a third unknown number, *c*

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Write down the integral of each of the following.

(i) $\int 4x + \frac{1}{x} \, dx$

[2 marks]

(ii) $\int cos\,(4x) \, dx$

[2 marks]

(iii) $\int e^{5x} \, dx$

[2 marks]

(iv) $\int \sqrt{x} - sin\,(0.5x) \, dx$

[4 marks]

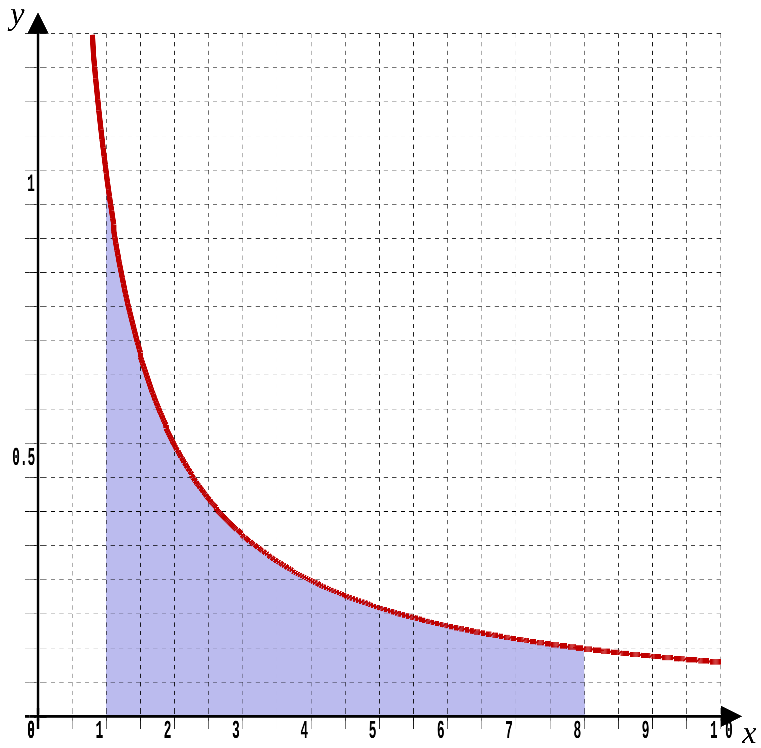
(v) $\int \left(1 + \frac{1}{x}\right)^2 dx$

[4 marks]

Top Tip : For each of the above, differentiating the answer should reverse the integration and get you back to the question. This is a handy way of checking any answer that you're not sure about.

Question 2

The graph is of the curve $y = \dfrac{1}{x}$

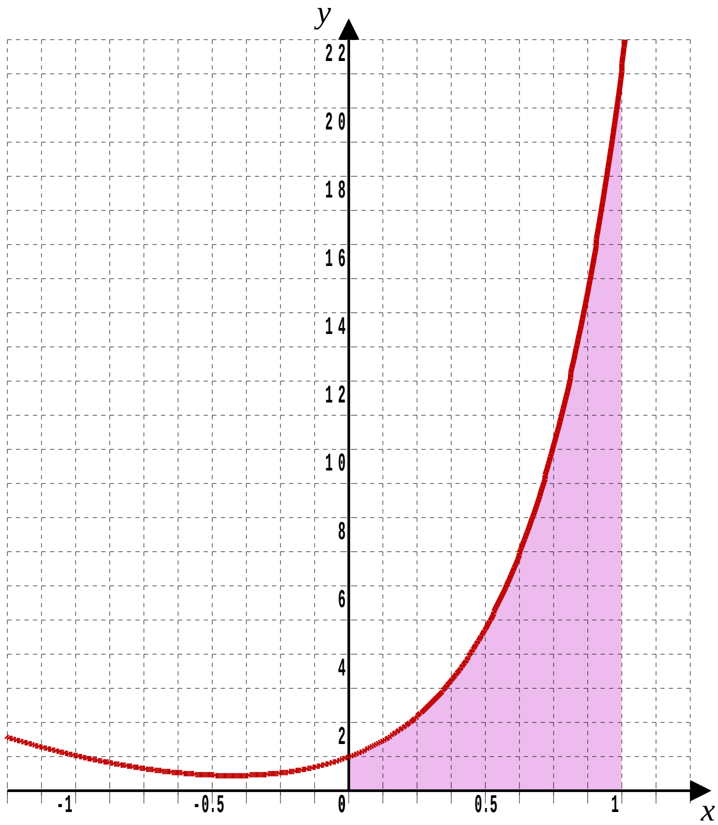


Use integration to show that the area shaded, bounded by the curve, the x-axis and the lines $x = 1$ and $x = 8$ is exactly $3\,ln\,2$

[5 marks]

Question 3

The graph is of the curve $y = e^{3x} + x^2$



Use integration to show that the area shaded, bounded by the curve, the x-axis, the y-axis and $x = 1$ is exactly $\frac{1}{3}\left(e^3\right)$

[5 marks]

Question 4

Find the exact area under the following curve between the bounding lines given

$$y = 9\cos(2x) \qquad x = 0, \qquad x = \frac{\pi}{4}$$

[5 marks]

Question 5

(i) Show that the area under the curve $y = 12\,e^{6x}$ between $x = 0$ and $x = 2$ is exactly $2\left(e^{12} - 1\right)$

[5 marks]

(ii) Give this area as a decimal correct to three decimal places.

[1 mark]

Question 6

Find the area under the curve between the bounding lines given

$$y = \frac{1}{9} sin\left(\frac{x}{2}\right) \qquad x = 0, \qquad x = \frac{\pi}{6}$$

Give your answer correct to 3 significant figures.

[5 marks]