Lesson 4

A-Level Pure Mathematics, Year 1
Additional Mathematics
The Algebra of Polynomials

4.1 Deeper Understanding

Suppose that $f(x) = x^2 + 7x + 10$
To find the factors of this using The Factor Theorem would involve testing from the list $(x \pm 1)$, $(x \pm 2)$, $(x \pm 5)$ and $(x \pm 10)$
An astute observer would notice that substituting in any positive value for x could not result in $f(x) = 0$ as there are no minus signs in the function.
So the list is halved in size !

To see if $(x + 1)$ is a factor test if $f(-1)$ is equal to zero or not,

$f(-1) = (-1)^2 + 7(-1) + 10$
$= 4$ As $f(-1) \neq 0$, $(x + 1)$ is not a factor, by the factor theorem

To see if $(x + 2)$ is a factor test if $f(-2)$ is equal to zero or not,

$f(-2) = (-2)^2 + 7(-2) + 10$
$= 0$ As $f(-2) = 0$, $(x + 2)$ is a factor, by the factor theorem

4.2 Why does the factor theorem work ?

Putting $x = -1$ into $x^2 + 7x + 10$ is the same as putting it into $(x + 2)(x + 5)$

$$\begin{aligned} f(x) &= x^2 + 7x + 10 \\ &= (x + 2)(x + 5) \\ \therefore f(-1) &= (-1 + 2)(-1 + 5) \\ &= 1 \times 4 \quad \leftarrow \text{ No factor “hit” so no zero} \\ &= 4 \end{aligned}$$

Putting $x = -2$ into $x^2 + 7x + 10$ is the same as putting it into $(x + 2)(x + 5)$

$$\begin{aligned} f(x) &= x^2 + 7x + 10 \\ &= (x + 2)(x + 5) \\ \therefore f(-2) &= (-2 + 2)(-2 + 5) \\ &= 0 \times 3 \quad \leftarrow \text{ Factor “hit” gives a zero} \\ &= 0 \end{aligned}$$

This shows that whenever $(x - a)$ is a factor, substituting a into the function $f(a) = 0$, which is The Factor Theorem ! □

Here is a reminder of exactly what the factor theorem says, which may make more sense having gained experience in using it and an understanding why it works.

The Factor Theorem

If, for a given polynomial function $p(x)$, $p(a) = 0$ (for some constant, a)

then $(x - a)$ is a factor of $p(x)$

4.3 Example

The function, $f(x) = x^4 + x^2 + kx$, where k is a constant, has a root $x = 1$

Fully factorise $f(x)$

Teaching Video : http://www.NumberWonder.co.uk/v9029/4.mp4

[5 marks]

4.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

By using the factor theorem, solve the equation

$$x^4 - 3x^2 + 2x = 0$$

[5 marks]

Question 2

Additional Mathematics Examination Question from June 2019, Q7 (OCR)

In this question you must show detailed reasoning

The equation $x^3 - 3x + k = 0$, where k is a constant, has a root at $x = 2$

Find the numerical value(s) of the other roots of this equation

[5 marks]

Question 3

Additional Mathematics Examination Question from June 2011, Q7 (OCR)

(a) Determine whether or not each of the following is a factor of the expression

$$x^3 - 7x + 6$$

You must show working

(i) $(x - 2)$

[2 marks]

(ii) $(x + 1)$

[1 mark]

(b) **(i)** Factorise the function $f(x) = x^3 - 7x + 6$

[3 marks]

(ii) Solve the equation $f(x) = 0$

[1 mark]

Question 4

Additional Mathematics Examination Question from June 2017, Q6 (OCR)

You are given that the equation $x^3 - x^2 - 10x + 6 = 0$ has two non-integer positive roots and one negative integer root.

(i) Using the factor theorem, find the negative root

[2 marks]

(ii) Hence solve the equation

[4 marks]

Question 5

Additional Mathematics Examination Question from June 2016, Q4 (OCR)

You are given that $f(x) = x^3 - x^2 + x - 6$

Show that

(i) $(x - 2)$ is a factor of $f(x)$

[1 mark]

(ii) the equation $f(x) = 0$ has only one real root

[4 marks]

Question 6

If the equations

$$x^3 + x^2 - ax - 2b = 0$$

$$3x^3 - 2x^2 - 3ax + 4b = 0$$

both have a solution $x = 2$, find a and b.

With these values of a and b show that the equation

$$x^3 + x^2 - ax - 2b = 0$$

also has a solution $x = -2$, and find the third solution.

[6 marks]

Question 7

Given that

$$f(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$$

find the values of $f(1)$, $f(-2)$ and $f(-3)$

It is given that $f(x)$ has two pairs of repeated factors, so that $f(x)$ may be expressed in the form $(x - a)^2 (x - b)^2$ where a and b are integers

Find an expression for $f(x)$ in the form $\left(px^2 + qx + r\right)^2$ where p, q and r are integers to be found.

[6 marks]