# Aperiodic engineering
# from combinatorics on words

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1. *A visual discovery*

The recent discovery of the aperiodic monotiles known as the *hat* and the *spectre* has sparked a surge of interest by the general public in exactly what it means to tile the Euclidean plane in an aperiodic fashion. Explaining a mathematical breakthrough to those without background knowledge can be difficult. However, in this case, communicating the ideas has been facilitated by the highly visual nature of what has been found, much to the delight of the editors of national newspapers and magazines, and also those who love to blog on social media. News of the first tile, the hat, broke in March 2023, with a second tile, the spectre, announced two months later. A festival, *hatfest*, took place a couple of months after that; the two day event featured lectures for established and aspiring mathematicians, a display of associated art, and an open-to-the-public evening lecture, all at the Mathematical Institute at the University of Oxford. Figure 1 shows the two tiles the centre of the festivities. To aperiodically tile the plane the hat requires the use of its mirror tile but the spectre, with a minor modification to its edges, is the first true monotile; so adjusted it, like the hat, can tile the plane aperiodically and not periodically, but without the need for its congruent mirror.

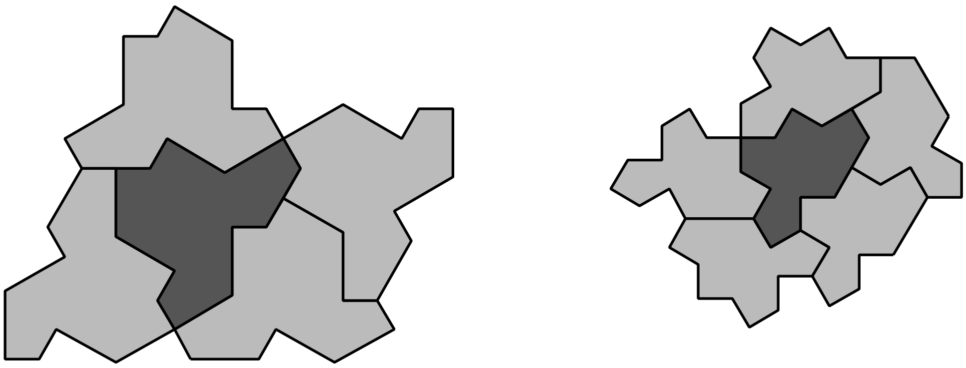

FIGURE 1: The hat (left) and spectre (right)

2. *Is aperiodicity of any practical use ?*

This article aims to provide an succinct initiation into the world of aperiodicity in a manner that relates it to the readily understood physical entities of tiling friezes and brick built walls. The approach taken to building some aperiodic structures is different to that which gives rise to the hat and the spectre but we will work our way towards being able to discuss and appreciate the extraordinary breakthrough they represent. The intention is to present big ideas alongside straight forward mathematics; mostly undergraduate matrix algebra. Consideration will be given to what sort of practical engineering applications aperiodic structures may be suited.

3. *Periodic*

Before progressing further, clarity is needed on the difference between periodic, non-periodic and aperiodic tilings. To keep things simple, our discussion will focus on linear tiling friezes, like that depicted in Figure 2. This frieze continues in the obvious way indefinitely to the left and to the right. Algebra is introduced by labelling each square *a*, and each rectangle *b*. Everyday experience means that Figure 2 is readily understood to be periodic, but let's focus on establishing this in a manner that will then help with the non-periodic and aperiodic situations.

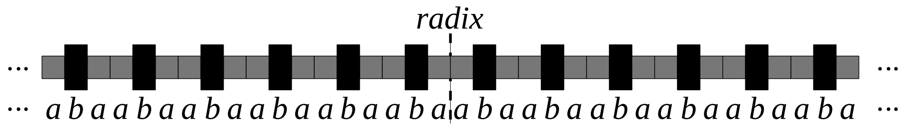


FIGURE 2: A periodic tiling frieze

If we were to overlay a sheet of thin, transparent plastic over the frieze and copy the pattern of squares and rectangles onto it, then there are many translations that will result in the plastic copy fitting exactly over the original. This is our test to determine if a frieze is periodic. We can construct this periodic frieze using an iterative substitution where we start with the *seed* given by *a* ¦ *a*, where ¦ is the radix point, much like the decimal point in a decimal fraction. Each iterative step replaces every occurrence of *a* with *aba* and each occurrence of *b* also with *aba*. The seed then grows the following sequence of *words*,

*a* ¦ *a*

*aba* ¦ *aba*

*abaabaaba* ¦ *abaabaaba*

*abaabaabaabaabaabaabaabaaba* ¦ *abaabaabaabaabaabaabaabaaba*

Our periodic frieze is the bi-infinite *fixed point* of an iterative process carried out indefinitely, initiated upon the seed which is word 0. Frequency information can be captured by writing down the *incidence matrix* of the substitution, in this case,

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

The 2 top left and the 1 bottom left record that at each iterative step *a* was replaced with 2 *a* and 1 *b* respectively. Similarly the adjacent column gives the frequencies of the substitution for each letter *b*. Consider the following;

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 18 & 18 \\ 9 & 9 \end{pmatrix}$$

This tells us that in word 3, each letter *a* in the seed has become 18 *a* and 9 *b*. Any *b* in the seed (there were none) would similarly have grown. When every entry in an incidence matrix is positive, or every entry in some integer power of that matrix is positive, then both the matrix and the substitution it represents are termed *primitive*. For a primitive matrix, *Perron-Frobenius Theory* states that there will be a unique positive largest eigenvalue, termed the Perron-Frobenius eigenvalue, $\lambda_{PF}$. In consequence, the relative frequencies are well defined and are given by the corresponding statistically normalized Perron-Frobenius eigenvector, $\boldsymbol{v}_{PF}$. For our periodic example, a chain of reasoning staring with the characteristic polynomial goes,

$$\lambda^2 - 3\lambda = 0 \Rightarrow \lambda_{PF} = 3 \Rightarrow \boldsymbol{v}_{PF} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This confirms the obvious result that in the infinite periodic word two-thirds of the letters are *a* and one-third are *b*. When the Perron-Frobenius eigenvalue of a primitive substitution is rational the tiling may or may not be aperiodic.

4. *Non-periodic but not aperiodic*

Figure 3 shows another bi-infinite tiling frieze, this time composed almost entirely of square tiles but for two that are rectangular. If, as before, we duplicate the frieze onto transparent plastic, there is no translation that can align the copy with the original other than applying no translation at all. Consequently this frieze is not periodic. However, because it is mathematically uninteresting, it is excluded from being classified as aperiodic. To be aperiodic, in addition to being non-periodic, it is required that the frieze have no arbitrarily large periodic part.

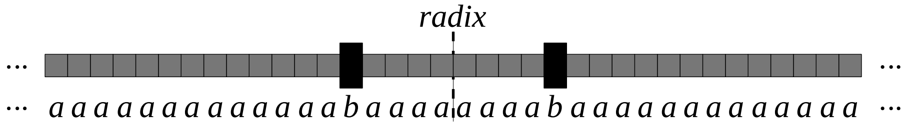


FIGURE 3: A non-periodic tiling frieze that is not aperiodic

To illustrate how such non-periodic yet also not aperiodic friezes are excluded using mathematics consider the substitution where *a* is replaced with *aba* and *b* with *bb*, this substitution rule then again acting iteratively on the seed *a* ¦ *a*,

*a* ¦ *a*

*aba* ¦ *aba*

*ababbaba* ¦ *ababbaba*

*ababbababbbbababbaba* ¦ *ababbababbbbababbaba*

As the iteration progresses further there is, in particular, an arbitrarily large periodic sequence of only the letter *b* at the centre of the left-sided and at the centre of the right-sided finite words. If the seed is labelled word 0, then word *n* has a sequence of *b* of length $2^{n-1}$. For the incidence matrix the calculation,

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ n\,2^{n-1} & 2^n \end{pmatrix}$$

shows the matrix will always have a zero top right. Therefore the substitution is not primitive. It is the requirement that the substitution be primitive that excludes friezes that are non-periodic but not aperiodic. The arbitrarily large periodic part, the failure to be primitive, causes the relative frequencies to not be defined.

5. *Aperiodic*

A straight forward example of an aperiodic tiling frieze is provided by the silver mean substitution, This replaces *a* with *aba* and *b* with *a*. Once again this acts on the seed *a* ¦ *a* in an iterative manner generating a fixed point. Here is the start,

*a* ¦ *a*

*aba* ¦ *aba*

*abaaaba* ¦ *abaaaba*

*abaaabaabaabaaaba* ¦ *abaaabaabaabaaaba*

Showing how the seed grows and the associated tiling frieze, given in Figure 4, do not provide insight into what makes the silver mean substitution so much more special than those considered previously, but the mathematics does.

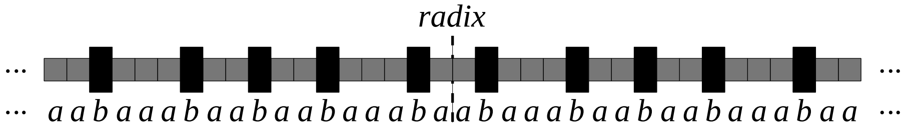


FIGURE 4: The silver mean tiling frieze

The incidence matrix is primitive, in spite of the lower right zero, because an integer power of the matrix exists with all positive entries.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ is primitive because } \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \text{ has all positive entries.}$$

The chain of reasoning now runs,

$$\lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda_{PF} = 1 + \sqrt{2} \Rightarrow \boldsymbol{v}_{PF} = \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 2 - \sqrt{2} \end{pmatrix}$$

The irrational eigenvalue allows for a lazy deduction that this substitution is aperiodic. This is thanks to a theorem proven by the late Uwe Grimm and Michael Baake in their classic book, *Aperiodic Order* (Volume 1), [1], page 89. The fact that the aperiodic tiling frieze of Figure 4 is clearly not random but also not radically different to the periodic frieze of Figure 2 gives a sense that this particular aperiodic substitution is a good example of being close to periodic, without actually being periodic.

## 6. *Building a brick wall*

With the foregoing basic ideas and techniques in place, thoughts can turn to an application of aperiodicity. Rather than intricate technicalities, we'll keep it simple, the focus instead being on enhancing intuition and considering the broader ideas that are motivating those working in this field. As a familiar structure, consider the standard periodically built brick wall shown in Figure 5. Beloved by the Victorians, many examples have, literally, stood the test of time. Given a firm foundation, they are particularly strong at supporting an evenly distributed force acting directly downward, such as from a roof.

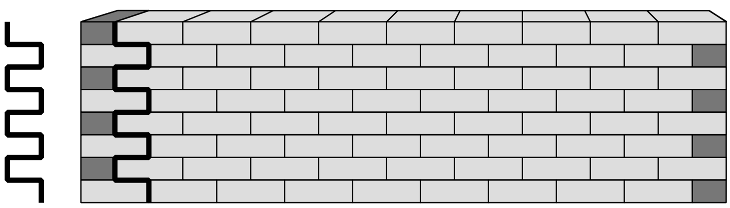

FIGURE 5: A wall built periodically with its profile *snake* shown separately, left

Figure 5 also shows what we will term the profile *snake* for the wall. Like the wall itself, this is periodic, and it is this that will be replaced with something aperiodic. As the wall will be built starting on the left and going right, only a right sided infinite word is needed, which is achieved with a seed of ¦ *a* and an iterative rule based upon the Thue-Morse substitution, $\theta_{TM}$, which replaces *a* with *ab* and *b* with *ba*. The first few Thue-Morse words are given in Table 1.

| $n$ | $TM_n = \theta_{TM}^n(a) \quad (a \to ab, b \to ba)$ | $\lvert a \rvert$ | $\lvert b \rvert$ | $\lvert TM_n \rvert$ |
|---|---|---|---|---|
| 0 | $TM_0 =$ ¦ $a$ | 1 | 0 | 1 |
| 1 | $TM_1 =$ ¦ $ab$ | 1 | 1 | 2 |
| 2 | $TM_2 =$ ¦ $abba$ | 2 | 2 | 4 |
| 3 | $TM_3 =$ ¦ $abbabaab$ | 4 | 4 | 8 |
| 4 | $TM_4 =$ ¦ $abbabaabbaababba$ | 8 | 8 | 16 |
| 5 | $TM_5 =$ ¦ $abbabaabbaababbabaababbaabbabaab$ | 16 | 16 | 32 |

TABLE 1 : The Thue-Morse substitution applied iteratively to an initial letter *a*.

Readers are invited to write down the primitive incidence matrix, and verify the intuitively obvious fact that the relative frequencies are 50% for *a* and 50% for *b*. Interestingly, the Perron-Frobenius eigenvalue is $\lambda_{PF} = 2$. As demonstrated previously, such integer eigenvalues leave open the possibility that the substitution is periodic. In support of this article a detailed three page proof that the Thue-Morse substitution is aperiodic is given in [2]. An aperiodic profile snake is generated by assigning a drawing rule to the Thue-Morse word as detailed in Table 2. The resulting profile snake and wall are shown in Figure 6.

| Symbol | Action |
|---|---|
| *a* | forward 0.5, turn right 90°, forward 0.5 |
| *b* | forward 0.5, turn left 90°, forward 0.5 |

TABLE 2 : The drawing rule

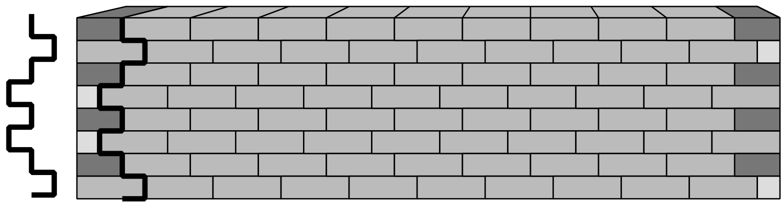

FIGURE 6 : A wall built periodically with its profile *snake* shown separately, left

In designing the aperiodic profile snake the silver mean substitution, although aperiodic, would not have worked as effortlessly as Thue-Morse because it contains the subword *aaa* which would have caused the snake to loop back on itself. To prove the Thue-Morse snake will not do this Definition 1 may be used.

*Definition 1* : The Thue-Morse Words by Concatenation

$$TM_n = TM_{n-1}\overline{TM_{n-1}} \text{ for } n \in \mathbb{Z},\ n \geqslant 1 \text{ with } TM_0 = a$$

For example,

$$\begin{aligned} TM_4 &= TM_3\overline{TM_3} \\ &= abbabaab\overline{abbabaab} \\ &= abbabaabbaababba \end{aligned}$$

*Theorem 1*: The Thue-Morse word when interpreted geometrically with *a* and *b* representing turns of 90° right and left respectively, never loops back on itself.

*Proof (Geometric Induction)*

The argument is inductive with a geometric flavour. As a basis for the induction consider $TM_3$ = *abbabaab* and observe that as a drawing, shown in Figure 7 (i), this comprises of pulses of unit depth to the left or right of a central vertical axis. It has turn accumulation two, meaning that, no matter where in the word we start, there are, geometrically speaking, never more than two turns to the right (or left) more than the turns to the left (or right). Furthermore, the entrance and exit directions are the same and at those points the turn accumulation is zero.

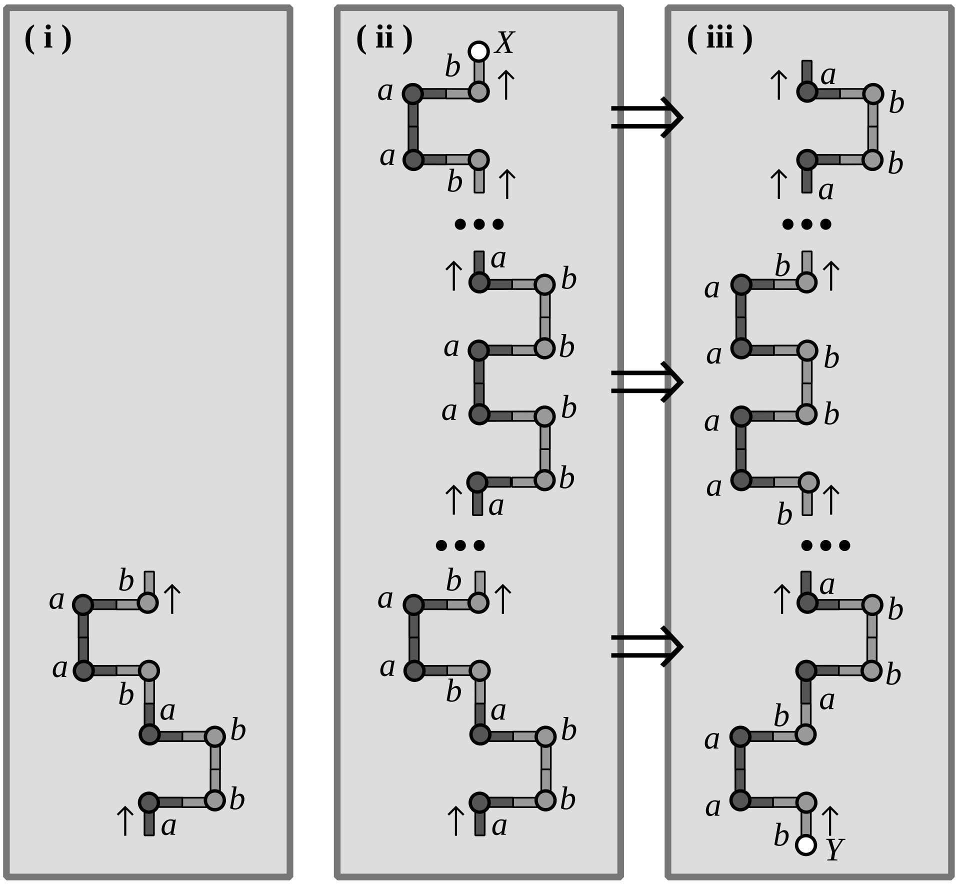


FIGURE 7 : Illustrations for the proof of Theorem 1

Now consider a word $TM_k$ and assume that it is composed of unit depth pulses to the left and right of a central vertical axis and so has turn accumulation two, as illustrated in Figure 7 (ii). Again, the entrance and exit directions are identical and at those points the turn accumulation is zero. From the definition given above of the Thue-Morse words, the next word, $TM_{k+1}$, is obtained by first forming $\overline{TM_k}$ which, geometrically speaking, is equivalent to a reflection of (ii) in its vertical central axis, thus forming (iii). This has caused all right turns to become left turns (and vice versa). Notice that (iii) is again composed of unit depth pulses to the left and right of a central axis and so again has turn accumulation two. The formation of $TM_{k+1}$ is completed by the concatenation of $TM_k$ with $\overline{TM_k}$ . Geometrically this corresponds to taking (iii) and translating it such that the end marked *Y* is placed over the end of (ii) marked *X*. Crucially, the combined drawing continues to have unit depth pulses to the left and right of a central axis and so again has turn accumulation two. Yet again, the entrance and exit directions are the same and at those points the turn accumulation is zero. Invoking the principle of mathematical induction we conclude that all Thue-Morse words when drawn in this manner will have unit depth pulses to the left and right of a central axis and have turn accumulation two. Consequently, the profile snake of infinite length never loops back on itself, or otherwise self-intersects. We have also demonstrated that the deviation to the left or the right of the central vertical axis is never more than one unit.

7. *Thue-Morse insight*

The key to understanding the Thue-Morse word is in realising it is cube-free. At a simple level this means, for example, that neither $aaa = a^3$ nor $ababab = (ab)^3$ can occur in the word. It's more profound than that. Take any piece of the Thue-Morse word, and that piece will never occur more than twice in succession. The piece will actually occur an infinite number of times throughout the infinite word (the word is said to be *repetitive*) but never more that once immediately after itself. Aperiodicity is then deduced from this fact.

8. *The hat in more detail*

Having seen how, from definition 1, the Thue-Morse aperiodic word was mechanically grown from a seed and an iterative rule, it's worth mulling over how the hat and the spectre aperiodic tilings arise from a distinctly different approach. Given the Thue-Morse word, there was nothing special about the tiles assigned to the letters *a* and *b*. To emphasise; the 90°-turn left and 90°-turn-right tiles where not in themselves what caused the aperiodicity. In stark contrast, for the hat and the spectre, it is the shape of the tile that is crucial; a local constraint (the shape of the tile) gives rise to a global property (aperiodicity).

A highlight of the July 2023 *hatfest* festival was when a few hundred congruent wooden copies of the hat were made available along with the challenge to the mathematicians attending the conference to start to tile the floor of the Andrew Wiles building aperiodically. The tiles had to be fitted together without gaps or overlaps. It proved to be difficult to make progress. This was because a hat tile does not come with a simple set of instructions of how it is to be placed with regards to those already there. It is all too easy to drift into a dead-end situation. A simple example of this is illustrated in Figure 8, where it is impossible to add more hat tiles to those already there without a resulting gap or overlap.

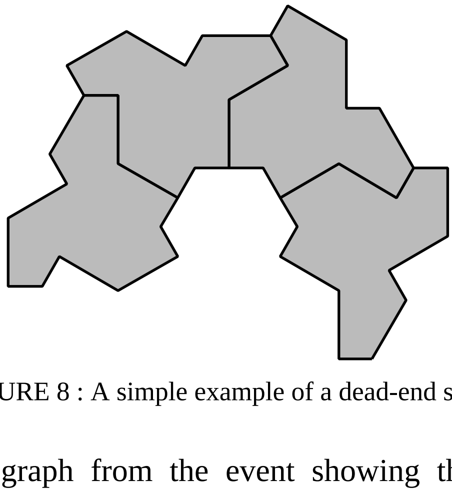

FIGURE 8 : A simple example of a dead-end situation

Figure 9 is a photograph from the event showing the state of play at the conclusion of the two day conference. Much time and effort from many attendees had gone into getting this far. What was set out was believed to be correct but, when under construction, there where moments when an inlet of the evolving island of tiles failed to resolve and lead to a dead end. Although it is known that the hat satisfies the essential conditions that,

- it can not tile the plane periodically
- it can tile the plane aperiodically

it is far from straight forward to physically tile a room's floor with a few hundred appropriately sized hat tiles. This raises intriguing questions about whether or not such a structure could arise in nature. How would a natural construction evolve that somehow 'knew' how to avoid a dead end? Quasicrystals provide the smoking gun that suggests it is not only mankind that can engineer such structures. Quasicrystals have a structure that is aperiodic in three dimensions. The discovery in the early 1980s of these substances that nature has somehow created is all the more extraordinary once one has experienced “hands on” how tricky it is to tile a floor with hat tiles.

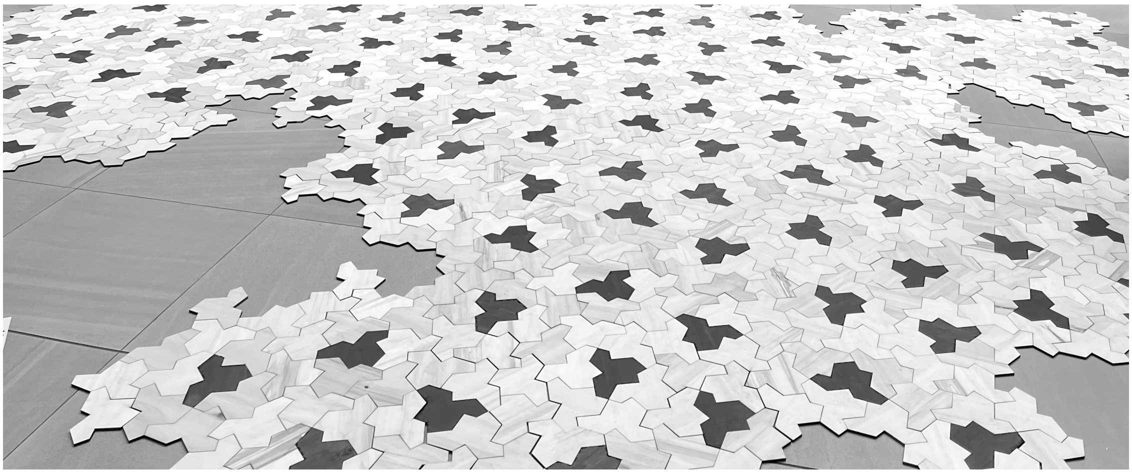

FIGURE 9 : A physical patch of the hat tiling on the floor of the Andrew Wiles building, Oxford

Returning to Figure 9, the dark tiles are those that are a mirror image of the light. The “almost” regular nature of their distribution illustrates that, as with our silver mean tiling frieze of Figure 4, this aperiodic structure is tantalising close to being periodic. The discovery of the hat tiling is a remarkable achievement; a suitable tile had to be found, it had to be realised that it had the potential to tile the plane aperiodically but not periodically, and then the suspicion confirmed mathematically. Further details of the discovery and subsequent mathematical analysis are easily found on the internet or by consulting the original papers announcing the find of the hat [3] and the spectre [4].

9. *Brick wall engineering properties*

Physical periodic structures are, perhaps, most vulnerable to unevenly distributed forces from a single direction, and forces from unexpected directions. One of their greatest weaknesses is to the unintended force, caused by, for example, an over enthusiastic gardener piling a large mound of earth against one side of a wall. Additionally, in practice, walls may have windows and doors. These can exacerbate any uneven force acting upon the wall. House surveyors know to look for structural movement around such openings. Old brick structures used lime mortar which is softer, more flexible and more porous than those built in modern times with the easier to use, faster setting, and greater load bearing properties of Portland cement. Victorian houses built with lime mortar are remarkably good at flexing to mitigate some of the effects of loading that has shifted over time; less so those with more modern Portland cemented walls. Mixing the two types of filler between the bricks in the same wall is a bad idea with many a lime mortar wall destroyed by being repointed with Portland cement. A possible advantage of the aperiodic wall is it will resist distortion more, especially when a force is applied to it from an unexpected, undesigned for, direction. In spite of the ubiquity of brick walls, a detailed analysis of their load bearing properties that treats the cement differently to the brick is relatively recent. This is typically accomplished using finite element analysis as, for example, described in [5]. The author would be interested should any reader have access to the capability to carry out a similar analysis on an aperiodic wall either experimentally or computationally. Our one-dimensional Thue-Morse wall is particularly simple and so suited to such an analysis. For a two-dimensional approach see, for example, Michal Dekking's 2012 paper [6].

## 10. *Is the future of engineering aperiodic ?*

Given that periodic brick built walls are tried and tested over long periods of time, to compete with them an aperiodic wall would have to be more than just marginally better. In this article that requirement has been balanced against the cautious approach of finding an aperiodic wall that is not drastically different to what is known to already work; more a refinement to existing practice than something radically new. Aperiodicity comes into its own for a large structure with lots of small component parts arranged aperiodically. For medium sized structures it may be that a well chosen non-periodic design is as good. Generally speaking, “random” is unsatisfactory and unpredictable although, as part of an iterative “random-then-analyse” cycle that keeps the best random configuration found to date, it may prove effective, especially given abundant computing power and a mechanical means of realising the best random structure found.

Applications to which aperiodic designs seem suited revolve around not knowing from which direction “something” comes. For example, the ideal electric razor needs to cope with cutting hair no matter what direction it moves through the hair. Current research at the Institut des Sciences de la Terre in France may reveal that an aperiodic array of sensors is better at detecting low-level seismic activity than the periodic configurations traditionally used. This is precisely because the wavefront from an earthquake or underwater implosion, for example, can arrive from any direction. A search on “aperiodic” through recently filed patents on the Espacenet database [7] reveals much activity at the level of nano-technology material science. Although this article has focussed on the low-tech brick wall, aperiodicity is certainly being vigorously investigated to see what it can offer as part of a high-tech future.

## 11. *Acknowledgements*

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