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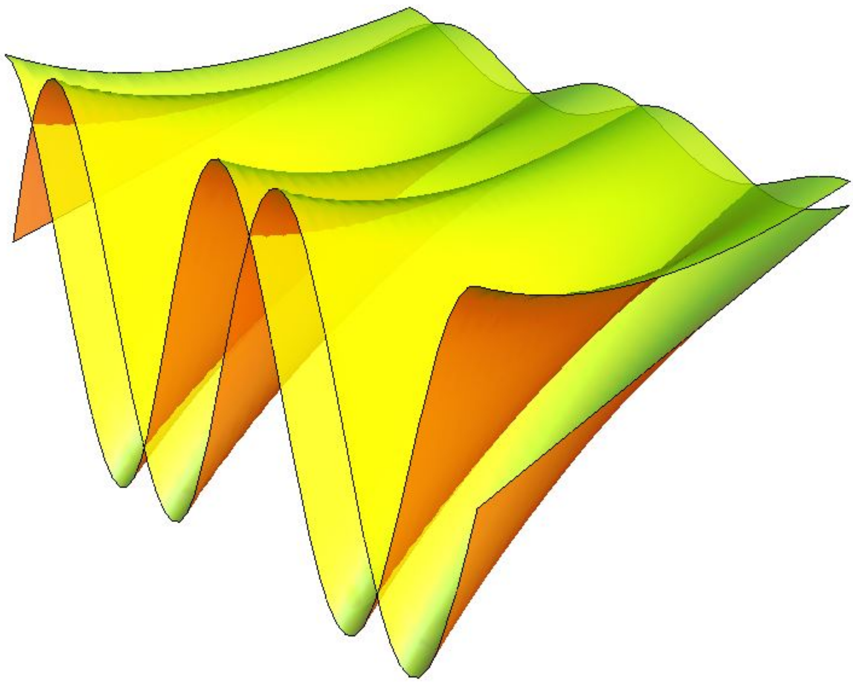

The Thue–Morse sequence is aperiodic: A proof

Martin Hansen

Although first investigated by the French mathematician Eugène Prouhet in 1851, the Thue–Morse sequence is named after the Norwegian, Axel Thue, who used it in 1906 as a foundation stone for the branch of mathematics *Combinatorics on Words*, and the American, Marston Morse, who applied it in 1921 to great effect in *differential geometry.* It can be defined in several ways. The following is particularly suited to introducing the sequence.

Definition : Generation of the Thue–Morse Word

The right-sided infinite Thue–Morse word may be generated in an iterative fashion by starting with an initial letter a and then repeatedly applying the substitution $\theta_{\mathcal{TM}}$ given by $a \to ab$, $b \to ba$. The finite n^{th} right-sided Thue–Morse word is defined as $\mathcal{TM}_n = \theta^n_{\mathcal{TM}}(a)$.

The table below gives the first few of the finite Thue–Morse words

n	$\mathcal{TM}_n = \theta^n_{\mathcal{TM}}(a)$ $(a \to ab,\, b \to ba)$	$\lvert a \rvert$	$\lvert b \rvert$	$\lvert \mathcal{TM}_n \rvert$
0	a	1	0	1
1	ab	1	1	2
2	$abba$	2	2	4
3	$abbabaab$	4	4	8
4	$abbabaabbaababba$	8	8	16
5	$abbabaabbaababbabaababbaabbabaab$	16	16	32

In the table, observe that each preceding word occurs at the start of every subsequent word. In other words, the infinite Thue–Morse word is the fixed point of an iterative process. What makes the Thue–Morse sequence interesting is the fact that it is clearly not a chaotic random jumble of the letters a and b. It seems to have pattern, but that pattern is hard to pin down.

The full bi-infinite Thue–Morse word extends indefinitely to both the left and the right. It has two remarkable properties that in combination are what earn it the label ‘aperiodic’. The first is that if a duplicate copy of the word is made, and translated left or right over the first, there is no position at which the two pieces align other than if no translation is applied at all. However, the following bi-infinite sequence also has this property,

$$\ldots aaaaaaaabaaaaaaaa \ldots .$$

This is an example of a sequence that is non-periodic but it is not aperiodic. To be aperiodic there is the additional requirement that the sequence contain no arbitrarily large periodic part.

Incidence Matrix and Perron–Frobenius Eigenvalue

Mathematically, this second requirement demands the substitution be primitive. To explain this property, we first need to establish what the incidence matrix of a substitution is. Let's look at an example where there are no parts that can be confused with each other. Consider the substitution,

$$a \to abaaaba, \qquad b \to abbb.$$

This has incidence matrix

$$\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$$

because, in the substitution, a is replaced with five a and two b whereas b is replaced with one a and three b. The matrix carries frequency information but the order of the letters within the substitution is lost. Undergraduate matrix algebra is now used to obtain the characteristic polynomial and the eigenvalues. For the above,

$$\lambda^2 - 8\lambda + 13 \;=\; 0 \qquad \text{with } \lambda = 4 \pm \sqrt{3}.$$

To be primitive the incident matrix (or a positive integer power of that matrix) must have all positive entries. The above matrix satisfies this requirement and so is the primitive matrix of a primitive substitution. Such a matrix has a largest positive eigenvalue (by Perron–Frobenius theory) called the Perron–Frobenius eigenvalue. For our example, $\lambda_{\mathrm{PF}} = 4 + \sqrt{3}$. In general, if this special eigenvalue is irrational, we immediately know that the substitution is aperiodic. This is because of Theorem 1, which formally tidies up the above discussion.

Theorem 1 : Aperiodic Proof (Baake and Grimm, 2013)

Let θ be a primitive substitution on the finite alphabet $\mathcal{A}_n = \{a_1, a_2, \ldots, a_n\}$ with incidence matrix $\mathbf{M}_\theta$, and let w be a bi-infinite word of θ. If the Perron–Frobenius eigenvalue of $\mathbf{M}_\theta$ is irrational, then w is aperiodic.

Proof

See *Aperiodic Order: Volume 1, A Mathematical Invitation*, by Michael Baake and Uwe Grimm, page 89. □

Unfortunately, Theorem 1 does not allow us to lazily deduce that the Thue–Morse word is aperiodic. Although the Thue–Morse substitution satisfies the requirement of the theorem that it be primitive, the incidence matrix gives rise to an integer eigenvalue rather than one that is irrational. To be specific,

$$\mathbf{M}_{\mathcal{TM}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda_{\mathrm{PF}} = 2.$$

The integer eigenvalue means that our desire to prove the Thue–Morse sequence is aperiodic is going to be a 'living on our wit and cunning' affair. Along the way we shall make use of another definition of the Thue–Morse words.

Definition : The Thue-Morse Words by Concatenation

$$\mathcal{TM}_n = \mathcal{TM}_{n-1}\,\overline{\mathcal{TM}_{n-1}}$$

for integer $n \geq 1$, with $\mathcal{TM}_0 = a$.

For example,

$$\mathcal{TM}_4 = \mathcal{TM}_3\,\overline{\mathcal{TM}_3} = abbabaab\overline{abbabaab} = abbabaabbaababba.$$

The Thue–Morse Language Table

We are now almost ready to start working through the steps that will result in a proof that the Thue–Morse word is indeed aperiodic. However, there is one more idea to be grasped before starting in earnest. It is to consider how many subwords of various lengths can occur in the Thue–Morse word. To help explain this here is the start of the right-sided infinite word once again,

$$abbabaabbaababbabaababbaabbabaab\ldots.$$

Looking carefully, notice that the subword *aaa* does not appear. This is an example of a cube. Whilst *abab* can be found (an example of a square) *ababab*, another cube, can not. A substitution's language table is the start of an effort to tabulate which subwords can occur. It can be drawn up by hand, but often software is used. Here is the start of such a table for the Thue–Morse word. This table will be useful later on.

length	1	2	3	4	5	6	7	...
	a	aa	aab	$aaba$	$aabab$	$aababb$	$aababba$	...
	b	ab	aba	$aabb$	$aabba$	$aabbaa$	$aabbaab$	...
		ba	abb	$abaa$	$abaab$	$aabbab$	$aabbaba$	...
$\mathcal{TM}$		bb	baa	$abab$	$ababb$	$abaaba$	$abaabab$	...
words			bab	$abba$	$abbaa$	$abaabb$	$abaabba$	...
			bba	$baab$	$abbab$	$ababba$	$ababbaa$	...
$a \to ab$				$baba$	$baaba$	$abbaab$	$ababbab$	...
$b \to ba$				$babb$	$baabb$	$abbaba$	$abbaaba$	...
				$bbaa$	$babaa$	$baabab$	$abbaabb$	...
				$bbab$	$babba$	$baabba$	$abbabaa$	...
					$bbaab$	$babaab$	$baababb$	...
					$bbaba$	$babbaa$	$baabbaa$	...
						$babbab$	$baabbab$	...
						$bbaaba$	$babaaba$	...
						$bbaabb$	$babaabb$	...
						$bbabaa$	$babbaab$	...
							$babbaba$	...
							$bbaabab$	...
							$bbaabba$	...
							$bbabaab$	...
№ of words	2	4	6	10	12	16	20	...

The Proof

This builds steadily over the next three pages as we establish three lemmas and define what it means to be strongly cube free. We then show that the (infinite) Thue–Morse word is strongly cube-free and so aperiodic.

Lemma 1 : Exclusion from $\mathcal{TM}$ of aaa and bbb

The subwords aaa and bbb cannot occur in the Thue–Morse word.

Proof (by contradiction)

Suppose by way of deriving a contradiction that the three letter subword $\ldots a\boldsymbol{a}a\ldots$ has occurred in $\mathcal{TM}$. Focus on the middle $\boldsymbol{a}$. We can argue that this $\boldsymbol{a}$ must have come from the iteration in a previous word of a letter b because if it had come from a letter a then one of the letters adjacent to $\boldsymbol{a}$ would be b, which neither is. But if it had come from a letter b then, again, one of the letters adjacent to $\boldsymbol{a}$ would be b, which neither is. The only conclusion is that the original assumption, that $\ldots aaa \ldots$ could occur in the Thue–Morse word is false. By a similar argument, $\ldots bbb \ldots$ can also not occur in $\mathcal{TM}$. □

Lemma 2 : Exclusion from $\mathcal{TM}$ of *ababa* and *babab*

The subwords *ababa* and *babab* cannot occur in the Thue–Morse word.

Proof (by contradiction)

Assume that the five letter subword $\ldots ababa \ldots$ has occurred in $\mathcal{TM}$. With a view to desubstitution this word can be placed into letter pair brackets in two ways, either as $\ldots (ab)(ab)(a \ldots$ or $\ldots a)(ba)(ba) \ldots$, which we consider in turn.

CASE 1 : $\ldots (ab)(ab)(a \ldots$

The first bracketed pair desubstitutes to a, as does the second. In order to desubstitute the third bracket the subsequent letter must be b. We then have the following.

$$\begin{array}{cccccl} \ldots & (ab) & (ab) & (ab) & \ldots & \\ & \downarrow & \downarrow & \downarrow & \text{desubstitution} & \\ \ldots & a & a & a & \ldots & \end{array}$$

However, from Lemma 1, it is known that $\ldots aaa \ldots$ can not occur and so we deduce that $\ldots (ab)(ab)(a \ldots$ can not occur.

CASE 2 : $\ldots a)(ba)(ba) \ldots$

The last bracketed pair desubstitutes to b, as does the bracket before. In order to desubstitute the first bracket the previous letter must be b. We then have the following.

$$\begin{array}{cccccl} \ldots & (ba) & (ba) & (ba) & \ldots & \\ & \downarrow & \downarrow & \downarrow & \text{desubstitution} & \\ \ldots & b & b & b & \ldots & \end{array}$$

However, from Lemma 1, it is known that $\ldots bbb \ldots$ can not occur and so we deduce that $\ldots a)(ba)(ba) \ldots$ can also not occur.

It has thus been shown that neither *ababa* nor *babab* are subwords of $\mathcal{TM}$. □

Lemma 3 : The *aa*, *bb* constraint on $\mathcal{TM}$ subwords

Any subword of the Thue–Morse word that contains five letters or more, must contain *aa* or *bb*.

Proof

With only two letters to play with, the only way to write down a subword of length five without *aa* or *bb* occurring is to alternate the occurrences of *a* and *b*. This can be done in two ways, either *ababa* or *babab*. However, Lemma 2 tells us that neither of these is legal. Therefore any subword of five letters must contain either *aa* or *bb*. Any subword of more than five letters must contain five letter subwords. Such subwords must contain *aa* or *bb* and therefore so must any subword of more than five letters. □

Definition : Cube-free words

For a non-empty finite word u, let u_0 and u_z denote the first and last letters of u, respectively.
A weak cube is a word of the form uuu_0 (or, equivalently, $u_z uu$).
A word w is strongly cube-free if it does not contain any weak cubes.

Theorem 2 : Strongly Cube-free

The Thue–Morse word is strongly cube-free.

Proof (by contradiction)

Let us assume the opposite of the claimed result, that the Thue–Morse word, $\mathcal{TM} = w_0w_1w_2\ldots$, contains at least one subword, u, such that uuu_0 is a subword of $\mathcal{TM}$. Consider the minimal length of u and denote that minimal length l. In other words, $|u| = l$. Clearly, u cannot be a single letter a or a single letter b for then uuu_0 would be *aaa* or *bbb* respectively, both of which, from Lemma 1, are illegal. Nor can u be either of the double letter subwords *aa* or *bb* for the same reason. So, the minimum length that u can be is two and, even then, only *ab* or *ba* are possibilities. However, if u was *ab* or *ba* then uuu_0 would be *ababa* or *babab* respectively, both of which, by Lemma 2 are illegal. So $|u| \geq 3$ and $|uuu_0| \geq 7$. This implies that uuu_0 can only be found in a Thue–Morse word of seven or more letters. The first such word is $\mathcal{TM}_3 = abbabaab$. Notice that the occurrence of *bb* in this word is at w_1 and *aa* at w_5. The manner of generating subsequent words using

$$\mathcal{TM}_n \;=\; \mathcal{TM}_{n-1}\overline{\mathcal{TM}_{n-1}}$$

means both that there will always be at least two *aa* or *bb* and that all such occurrences are at positions w_x, where x is an odd number. Now consider the parity of l.

CASE 1 : l is odd

If $|u| = 3$, then we can draw up the following table of u and uuu_0 and show by desubstitution that none of the six possible uuu_0 can be in the Thue–Morse word. Alternatively it can be seen by an inspection of the Thue–Morse language table, given earlier, that none of these uuu_0 are legal subwords.

u	uuu_0
aab	$aabaaba$
aba	$abaabaa$
abb	$abbabba$
baa	$baabaab$
bab	$babbabb$
bba	$bbabbab$

For $|u| \geq 5$, we know from Lemma 3 that u contains aa or bb at least twice (as indeed does $\mathcal{TM}$). Where they occur in $\mathcal{TM}$ (and there will be an infinite number of such occurrences) must be at odd positions. Thus the distances between all occurrences of aa and bb are even. Crucially, one of these distances must be l. However, we are looking at the case where l is odd and so this contradicts that assumption. To summarise, the assumption that l is odd has led to a contradiction and so that assumption cannot be correct.

CASE 2 : l is even

Recall that in the Thue–Morse word deleting every other letter in an odd position results in the Thue–Morse word. So if a subword $u = w_0w_1w_2 \ldots w_{l-1}$ (of even length) gives a uuu_0 in $\mathcal{TM}$ then so too must $u' = w_0w_2 \ldots w_{l-2}$. This, however, is a shorter word, in contradiction to l being minimal.

In overall conclusion, the Thue–Morse word is strongly cube-free. □

Corollary : The Thue–Morse Aperiodicity

The Thue–Morse word is aperiodic.

Proof

This follows immediately from Theorem 2. □

Intuitive Understanding

The proof tells us that, given any piece whatsoever (of any finite length) of the Thue–Morse word, that piece will never occur more than twice in succession. It may well occur singly or as a pair an infinite number of (spaced out) times in the word, but that is another story.

Acknowledgement

An outline sketch of this proof was suggested by the late Uwe Grimm when I knew him at the Open University in 2021. Discussions in 2023 with Dr Nicholas Korpelainen and Dr Petra Staynova at the University of Derby further added to the mix of ideas that have now crystallised in the proof presented.

Problem 314.1 – Cosines

David Sixsmith

The cosine formula

$$a^2 \;=\; b^2 + c^2 - 2bc\cos A, \tag{1}$$

where a, b, c are the sides of a triangle and A is the angle opposite the side of length a, is well known. Equation (1) can be written in a symmetric 'Pythagorean' form

$$a^2 \;=\; (\alpha b + \beta c)^2 + (\beta b + \alpha c)^2. \tag{2}$$

An initial problem it to find α and β as simple functions of the angle A. A more difficult problem is to find a direct geometric justification for equation (2), in other words, a derivation that does not use the cosine formula. (I have not had success with the latter.)

Problem 314.2 – Rational eigenvalues

Tony Forbes

For which positive integers a does the matrix

$$M \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & b & c \end{bmatrix}$$

have non-zero determinant and rational eigenvalues for some positive integers b and c?

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03.02

Front cover Real and imaginary parts of $E(x + iy)$; see page 10.