Shrewsbury

CO-EDUCATIONAL BOARDING & DAY SCHOOL

Arnold Hagger Mathematics prize

Question paper

3rd February 2025

Surname: ____________________ First name: ______________

House: ______________________ Year group: ______________

Instructions

- Use black ink or ball-point pen
- **Answer as many questions as you can in the time**
- Answer the questions in the separate booklet provided.
- Marks are obtained for the quality and elegance of your solution. Answers which are not fully justified will not gain fully credit
- The total mark for this paper is 100. There are 15 questions.
- A calculator may be used but will not be particularly useful. An answer provided via a calculator where no logic or method is shown on paper would not expect to achieve marks on any question.

Question:	1	2	3	4	5	6	7	8
Points:	5	5	5	5	5	5	5	10
Score:								

Question:	9	10	11	12	13	14	15	Total
Points:	5	5	5	10	10	10	10	100
Score:								

1. Two real numbers, x and y, satisfy the equation
$$x^2 + y^2 + 3xy = 2025$$
What is the maximum value of xy? [5]

2. A conference has 47 people attending. One woman knows 16 of the men who are attending, another knows 17, and so on up to the last woman who knows all the men who are attending. Find the number of men and women who are attending the conference. [5]

3. The sum of the numbers $3m - 4, 3n - 4$ and $3p - 4$ is 2025. Find the sum of the number $4m - 3, 4n - 3$ and $4p - 3$. [5]

4. Prove that if you choose any five positive integers, there will be three which sum to make a multiple of 3. [5]

5. How many prime numbers between 10 and 99 remain prime when the order of their two digits is reversed? [5]

6. Prove that
$$\frac{1}{\frac{1}{a}+\frac{1}{b}} + \frac{1}{\frac{1}{c}+\frac{1}{d}} \leq \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}}$$
for positive real a, b, c and d [5]

7.

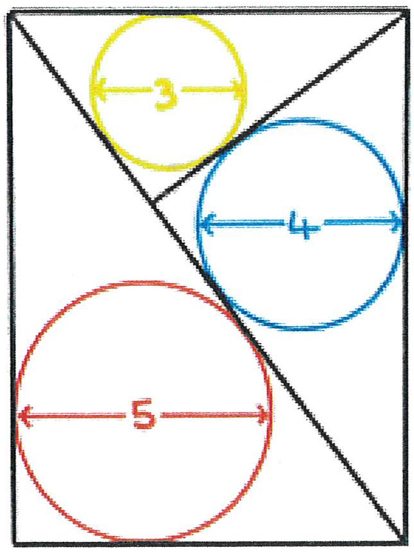

Figure 1

Figure 1 shows a rectangle divided into three triangles. The triangles each have an inscribed circle with the diameters shown. What is the area of the rectangle? [5]

8. 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

   (a) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning carefully. [4]

   (b) How many five-digit numbers are there whose digits sum to 39? [6]

9. Ten people are sitting around a round table. The sum of £10 is to be distributed among them according to the rule that each person receives one half of the sum that their two neighbours receive jointly. How many ways are tere to distribute the money? Prove your answer. [5]

10. Tom plays a game with Tim, they keep playing until one of them has two more wins than the other, at which point they are declared the big winner. Tom has a 0.6 chance of winning each small game. Tom and Tim have been playing the game, but there is no big winner yet. What is the chance that Tom is the big winner? [5]

11. A large, solid cube is built using 216 small cubes. 180 of the cubes are red and 36 are blue. The large cube is formed by arranging the cubes in such a way that the exposure of the red cubes to the outside is minimised. Calculate the percentage of exposed area that is red, giving your answer to the nearest integer. [5]

12. Mr Muddle has a total of n socks in a large drawer, of which x are blue. The remaining socks are red.
    He chooses two socks at random from the drawer without looking at them. The probability he gets matching socks is exactly $\frac{1}{2}$.

    (a) Find an expression for x in terms of n. [6]

    (b) Hence show that n must be a square number, and that the numbers of blue and red socks are consecutive triangular numbers. [4]

13.

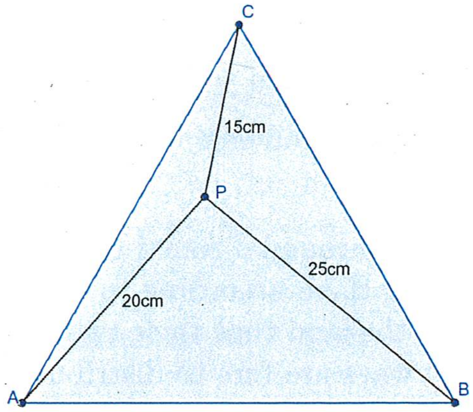

Figure 2

Figure 2 shows an equilateral triangle ABC with the length of some internal lines. Calculate the length of the side of the triangle. [10]

14. Let $S = \{a_1, a_2, ..., a_n\}$ where a_i are different positive integers. The sum of the elements of each non-empty propoer subset of S is not divisible by n. Show that the sum of all elements of S is divisible by n.
Note that a proper subset of S consists of some, but not all, of the elements of S [10]

15. On Wednesday 1st January 2025, Anna buys one book and one shelf. For the next two years, she buys one book every day and one shelf on alternate Thursdays, so she next buys a shelf on 15th January 2015. On how many days in the period Wednesday 1st January 2025 until (and including) Thursday 31st December 2026 is it possible for Anna to put all her books on all her shelves, so that there is an equal number of books on each shelf? [10]