Shrewsbury

CO-EDUCATIONAL BOARDING & DAY SCHOOL

Arnold Hagger Mathematics prize

Question paper

8th February 2024

Surname: ______________________		First name: ______________	
House: ________________________		Year group: ______________	

Instructions

- Use black ink or ball-point pen
- Answer as many questions as you can in the time
- Answer the questions in the separate booklet provided.
- Marks are obtained for the quality and elegance of your solution. Answers which are not fully justified will not gain fully credit
- The total mark for this paper is 100. There are 19 questions.
- A calculator may be used but will not be particularly useful.

Question:	1	2	3	4	5	6	7	8	9	10
Points:	5	3	5	4	3	4	4	7	4	4
Score:										

Question:	11	12	13	14	15	16	17	18	19	Total
Points:	7	4	5	6	5	12	4	4	10	100
Score:										

1. Aaron recently noticed that this time next year he will be twice as old as Noraa
   Mr N noticed that when he is 66 in the future, the combined future ages of Aaron and Noraa will also then be 66.
   Also, 6 years from now Mr N's age will be a multiple of the sum of Aaron and Noraa's then
   How old are Aaron, Noraa and Mr N now? [5]

2. Prove that all primes greater than 2 can be written as the difference of two squares. [3]

3. Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors. [5]

4. Here is a diagram of a black circle inside a white square and a diagram of a small grey square inside a white square.
   The two white squares are the same size.

   The sides of the square are tangents to the circle at the points E, F, G and H. W, X, Y and Z are the midpoints of AB, BC, CD and DA respectively.

   Given that $AB = 8x$, work out the ratio of the area of the black circle to the area of the white square to the area of the grey square. Give your answer in its simplest form. [4]

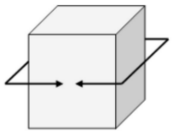
5. Prove that the product of four consecutive integers is always one less than a perfect square. [3]

6. A fly trapped inside a cubical box of side length 1 metre decides to relieve its boredom by visiting each corner of the box. It will begin and end in he same corner and visit each of the other corners exactly once. To get from a corner to any other corner it will either fly or crawl in a straight line. What is the maximum possible length, in metres, of its path. Give your answer in exact form. [4]

7. During a football match a Mr Ogan saves 33% of goals. After saving one more shot, his save rate rose to 40%. How many more shots on target does he need to save to raise his save rate to 50%? [4]

8. Given that $y = \dfrac{x}{x + \dfrac{x}{x+y}}$, find the range of values of x that do not give a real value for y [7]

9. For any given cuboid, it is possible to measure up to three different perimeters. For example, one perimeter could be measured this way



Given that cuboid A has perimeters 12,16 and 20, and cuboid B has perimeters 12, 16 and 24, which cuboid has the greatest volume. [4]

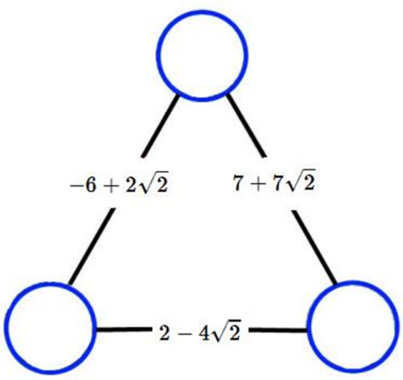
10. Mr An wants to sum a lot of numbers. He wants to write down every possible combination of the digits 1,2,3 and 4 and then sum them all. What would his answer be? [4]

11. Prove the following statements

   (a) The sum of five consecutive numbers is always a multiple of 5 [3]

   (b) If n is even, then the sum of n consecutive numbers is always a multiple of $\frac{n}{2}$ but not of n [4]

12. Mr Gan designs a multiple arithmagon.
He declares that a multiple arithmagon is a shape where the number on each edge is the product of the numbers at the adjacent vertices.
Work out the numbers that belong in the circles to make this multiplication arithmagon correct.



[4]

13. Prove that every term in the infinite sequence

$$18, 108, 1008, 10008, ...$$

is divisible by 18 [5]

14. Find all real solutions to this equation

$$(2 - x^2)^{x^2 - 3\sqrt{2}x + 4} = 1$$

You should demonstrate clearly how you know you have found all the solutions. [6]

15. Mr Rogan asks Mrs Nagor *"How many children do you have and what are their ages?"*
Mrs Nagor replies *"I have three children, the product of their ages is 36, the sum of their ages is equal to the address of the house next door"*
Mr Rogan looks at the house number next door, comes back and says *"I need more information"*
Mrs Nagor replies *"I'm afraid I have to go, my oldest child is calling me"*
Mr Rogan now knows the ages of all three children. What are they? [5]

16. Mr Brogan is investigating four numbers, a, b, c, d which all lie between -5 and 5. Mr Brogan puts some further constraints on the numbers:

$$5 < a + b < 10 \qquad \text{and} \qquad -10 < c + d < -5$$

Given this information, can you deduce further information about the following inequalities

(a) $?? < a + b - c - d < ??$

(b) $?? < a - c < ??$

(c) $?? < a - c + d - b < ??$

(d) $?? < abcd < ??$

(e) $?? < \frac{|a|+|c|}{2} - \sqrt{|ac|} < ??$

[12]

17. What is the value of $\sqrt{\dfrac{1}{2^6} + \dfrac{1}{6^2}}$. Marks are awarded for demonstrating your answer, not for the final answer. [4]

18. The ratio of boys to girls at a school disco is 9:10
An extra 17 boys arrive and the ratio changes to 8:7.
How many girls are there at the disco? [4]

19. *Note:* $a! = a \times (a-1) \times (a-2) \times ... \times 2 \times 1$

Find all solutions in positive integers a, b, c to

$$a!b! = a! + b! + c!$$

[10]