Arnold Hagger

Mathematics Prize

26th January 2022

90 minutes

- Answer as many questions as you can.

- All answers must be written in this question paper.

- There are 100 marks altogether.

- You are not necessarily expected to finish the paper. Producing a few complete, elegant solutions is better than doing scraps from each question.

- A calculator may be used in any question, but will not be very useful.

- Standard geometrical instruments are also allowed.

Good luck! JCA

1) Four donkeys take a Mechanics exam. Their scores (out of 100) have a range of 6, a mean of 7, a median of 8, and a mode of 9. What are the four scores?

[3 marks]

2) When Stewie bought a water pistol, he received £ 5.10 change, all in coins. He noticed that for each coin in his change, there were exactly two other coins of the same value. What was the coin of lowest value in Stewie's change?

[3 marks]

3) Sally the Snail travels back and forth between the Darwin and Philip Sidney statues. Her first journey takes 1 full day, and each subsequent journey takes twice as long. (Thus, the second journey takes 2 days, the third journey takes 4 days, and so on.)

Given that she sets off at noon on Monday, on which day of the week will she finish her sixth journey?

[3 marks]

4) A statue weighs 30 tons plus a third of its weight. How much does it weigh altogether? (Be careful…)

[3 marks]

5) Twelve people from 4 suits gather at a party:

- the King, Queen and Jack of Spades;
- the King, Queen and Jack of Hearts;
- the King, Queen and Jack of Diamonds;
- the King, Queen and Jack of Clubs.

Everyone shakes hands once with everybody else, except that no one shakes hands with someone of the same suit. How many handshakes are there altogether?

[3 marks]

6) A photograph fits perfectly on a 16:9 widescreen monitor, filling up the entire rectangular screen. The same photograph is then displayed on an older 5:4 monitor, filling up as much of the screen as possible without losing any of the image.

What is the exact proportion of the area of the older screen that is not used?

[4 marks]

7) Mr Fussy likes his marmalade to be completely smooth (no lumpy bits) and entirely orange (no yellow bits).

One morning he opens a new jar. To his dismay, he finds lumpy bits spread out at random, taking up 20% of the marmalade. He also notices yellow bits (which may or may not be lumpy) spread out at random, taking up 35% of the marmalade.

Once he has carefully weeded out all the bits he doesn't like, what percentage of the marmalade will be left for him to eat?

[4 marks]

8) Express $\left(\sqrt[3]{3}\right)^{\sqrt[3]{3}}$ in the form 3^{3^n} for a suitable value of n.

[4 marks]

9) The positive real numbers a, b and c satisfy:

$$ab = 90, \qquad bc = 5, \qquad ca = 2$$

Calculate the value of $a + b + c$.

[5 marks]

10) A newspaper consists of a number of sheets folded in half, each providing 4 pages. For example, a newspaper with 3 sheets would have 12 pages altogether: pages 1, 2, 11, 12 would be on the outer sheet, pages 3, 4, 9, 10 on the middle sheet, and pages 5, 6, 7, 8 on the inside sheet.

If one sheet of a newspaper contains page 36 and at least one page in the seventies, how many pages could the entire newspaper have altogether?

[5 marks]

11) Two points are chosen at random on the circumference of a circle. What is the probability that the length of the chord between the points is longer than the radius of the circle?

[5 marks]

12) Miss Take (who is careless) can mark a pile of Arnold Hagger exam scripts in 3 hours. Mr Page (who is inattentive) only needs 2 hours to mark a pile of scripts.

Dr Mark (who is ungenerous) joins the other two teachers. When all three work together, they can mark a pile of scripts in 36 minutes.

Calculate how long it takes Dr Mark to mark a pile of scripts by himself. Give your answer in hours and minutes.

[6 marks]

13) All the positive whole numbers are written in a grid, starting from 1, and arranged in an infinite clockwise spiral:

| 26	27	28	29	30	31
25	10	11	12	13	32
24	9	2	3	14	33
23	8	1	4	15	34
22	7	6	5	16	35
21	20	19	18	17	36
	...	39	38	37	

What are the eight numbers that surround 2022?

[6 marks]

14) A regular pack of 52 playing cards is shuffled, and all the cards are dealt in a line.

What is the exact probability that all four aces will appear before the first king **and** all four kings will appear before the first queen?
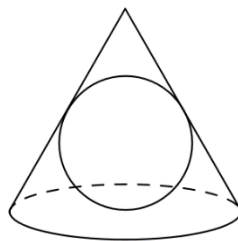
[6 marks]

15) The diagram below shows a quarter-circle inside a rectangle of width x and height y. The radii of the quarter-circle lie along adjacent edges of the rectangle, and the arc touches a diagonal of the rectangle.
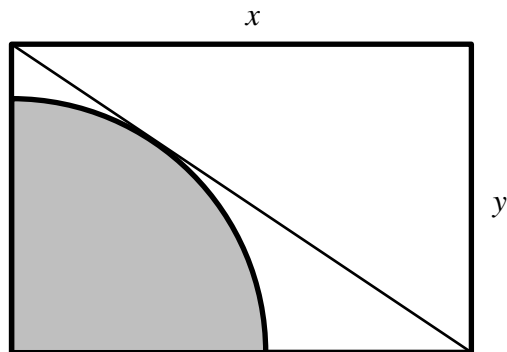
Find and simplify an expression in terms of x and y for the proportion of the rectangle which is occupied by the quarter-circle.
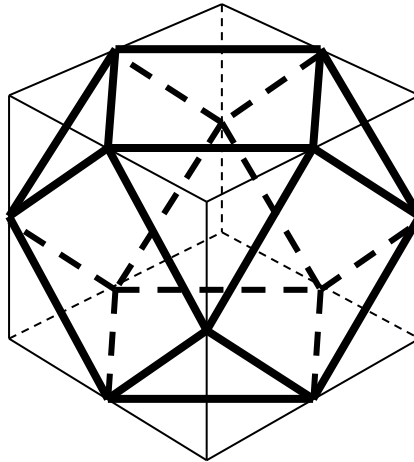
[7 marks]

16) A sphere of radius 1 unit just fits under a cone whose diameter and slant height are equal.

Calculate the exact volume of the cone, leaving π in your answer.

[7 marks]

17) The bold lines in the diagram show a regular **cuboctahedron**. This solid is formed by joining up the midpoints of all pairs of adjacent edges of a cube.

Given that the surface area of the cuboctahedron is 1 m^2, calculate the exact surface area of the outer cube. Give your answer in its simplest form.

[8 marks]

18) Find **all** solutions of the equation below, where x and y are integers.

$$\frac{2}{x} + \frac{73}{y} = 2$$

[8 marks]

19) Hexadecimal numbers are written in **base 16**, using these sixteen digits:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.

For example, the hexadecimal number 6AC is equal to the following decimal number:

$$(6 \times 16^2) + (10 \times 16) + 12 = 1708$$

How many hexadecimal numbers from 1 to FFFF FFFF FFFF FFFF inclusive are palindromes? Give your answer in hexadecimal form.

(A palindrome is a number which reads the same forwards as backwards, and which does not start with a zero. For example, 3F3 and 204402 are both hexadecimal palindromes.)

[10 marks]

+++++ **The End** +++++