Arnold Hagger

Mathematics Prize

21st January 2015

90 minutes

- Answer as many questions as you can.

- All answers must be written in this question paper.

- There are 100 marks altogether.

- You are not necessarily expected to finish the paper. Producing a few complete, elegant solutions is better than doing scraps from each question.

- A calculator may be used in any question, but will not be very useful.

- Standard geometrical instruments are also allowed.

Good luck!   JCA

1) Five hamsters take a Mathematics exam. Their scores (out of 100) have a range of 5, a mean of 6, a median of 7, and a mode of 8. What are the five scores?

[3 marks]

2) The White Rabbit has an appointment to see the Red Queen at 10 a.m. every day of the week. On Monday he arrives 32 minutes late. Every day he hurries more and more, and manages to halve the time he arrives late each day. On which day of the week will he arrive 15 seconds late?

[3 marks]

3) When Cartman bought a tricycle, he received £ 2.50 in change. He noticed that for each coin in his change, there was exactly one other coin of the same value. What was the coin of lowest value in Cartman's change?

[3 marks]

4) The four kings of clubs, diamonds, hearts and spades and their respective queens are having an arm-wrestling competition. Everyone must wrestle everyone else, except that no king may wrestle his own queen. How many arm-wrestles take place altogether?
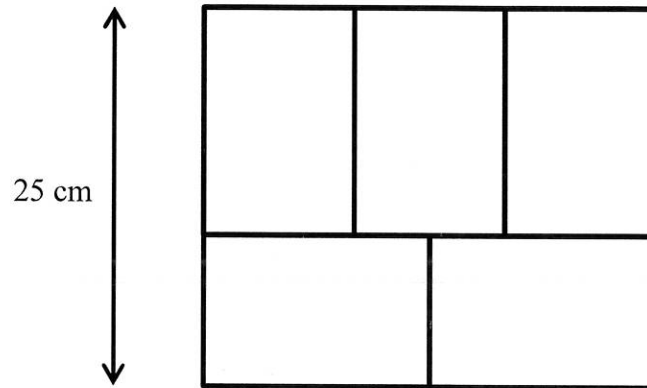
[3 marks]

5) A statue weighs 1000 kg plus half its weight. How much does it weigh altogether?

[3 marks]

6) A television programme has an image which fits perfectly on a widescreen TV with a 16:9 rectangular screen. If the same programme is shown on an old TV with a 4:3 screen, what proportion of the area of this screen is not used?

[4 marks]

7) Five identical rectangles fit together to make a large rectangle as shown below.

25 cm

Calculate the total area of the large rectangle.

[4 marks]

8) Mr Fussy likes his marmalade to be completely smooth (no lumpy bits) and entirely orange (no yellow bits). What a fussy old fusspot!

One morning he opens a new jar. To his dismay, he finds lumpy bits spread out at random, taking up 30% of the marmalade. He also notices yellow bits (which may or may not be lumpy) spread out at random, taking up 40% of the marmalade.

Once he has carefully weeded out all the bits he doesn't like, what percentage of the marmalade will be left for him to eat?

[4 marks]

9) The positive real numbers a, b and c satisfy:

$ab = 2$, $\qquad bc = 24$, $\qquad ca = 3$

Calculate the value of $a + b + c$.

[5 marks]

10) A newspaper consists of a number of sheets folded in half, each sheet providing 4 pages. For example, a newspaper with 3 sheets would have 12 pages altogether: pages 1, 2, 11, 12 would be on the outer sheet, pages 3, 4, 9, 10 on the middle sheet, and pages 5, 6, 7, 8 on the inside sheet.

If one sheet of a newspaper contains page 28 and at least one page in the sixties, how many pages could the entire newspaper have altogether?

[5 marks]

11) Willy Wonka's bathtub has three taps: a dark chocolate tap, a milk chocolate tap, and a white chocolate tap. When only one tap is turned on full, it takes 3 minutes to fill the bath with the dark tap, 4 minutes with the milk tap, and 6 minutes with the white tap.

How long does it take to fill Wonka's bath if all three taps are turned on full? Give your answer in minutes and seconds.

[5 marks]

12) In an unusual law court, a defendant may request one of two possible juries: a three-man jury or a one-man jury.

In the three-man jury, two members each have probability p of independently making the correct decision; the third member flips a fair coin for each decision. The majority verdict of the three members is used.

The one-man jury has one member who has probability p of making the correct decision.

Which jury has the higher probability of coming to the correct decision?

[5 marks]

13) A dot is placed at the centre of each face of a cube. A line is drawn between each pair of dots lying on adjacent faces of the cube. The resulting solid formed by these lines is a regular octahedron, as shown in the diagram.
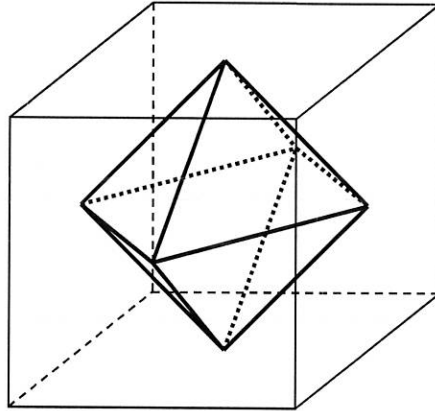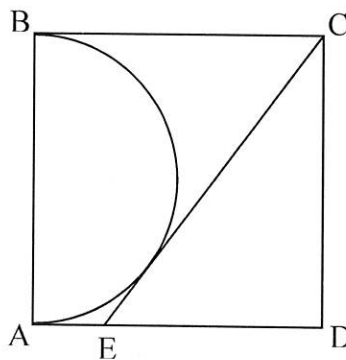

What proportion of the volume of the cube is taken up by the octahedron?

[6 marks]

14) A semicircle lies inside a square ABCD of side 2 units, with its diameter occupying the edge AB of the square.

Point E lies on edge AD so that the line CE is a tangent to the semicircle.

Find the area of triangle CDE.

[7 marks]

15) Let the binary operation $*$ be defined by: $a*b = \dfrac{a+b}{2}$.

For example, $3*5 = \dfrac{3+5}{2} = 4$.

   a) Describe in words what $*$ effectively calculates for two numbers a and b.

   b) Find a condition involving a, b, c such that $(a*b)*c = \dfrac{a+b+c}{3}$.

   c) Explain briefly why **no** expression (however long or complicated) involving only a, b, c, $*$ and brackets, can be equal to $\dfrac{a+b+c}{3}$ for **all** values of a, b, c.

   d) When does $(a*b)*c = a*(b*c)$?

[8 marks]

16) a) Given that $x+y$ is a factor of $x^3 + y^3$, factorise $x^3 + y^3$.

   b) **Using your result in (a)**, find all the factors of 1027.

   c) Prove carefully that there is only one prime number which can be expressed as the sum of two cubes of positive integers.

[8 marks]

17) Two spheres of **radii** a and b lie inside an upright cylinder of **diameter** d. The spheres touch the base of the cylinder, the inside curved surface of the cylinder, and each other.

   a) Find an expression for d in terms of a and b.

   b) Hence show that $\sqrt{d} - \sqrt{a} - \sqrt{b}$ is a constant, which you should determine.

[8 marks]

6

18) Find **all** solutions of the equation below, where x and y are integers.

$$\frac{14}{x} + \frac{7}{y} = 2$$

[8 marks]

19) In this question, a *run* is a block of cards which are all the same colour. For example, ■■□□□■ has 3 runs, and ■□■□■□ has 6 runs.

Dr Octopus has 8 black cards and 8 white cards. He shuffles all the cards together, and deals them in a row. What is the probability that there are exactly 8 runs? Give your answer as a fraction.

[8 marks]

+++++ **The End** +++++