Name	Set

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Mathematics Competition

2011

This paper carries a total of 100 marks.

Answer as many questions as you can. Do not spend too long on any one question, if you become stuck.

The marks for each question indicate the simplicity or intricacy of that question.

Answers without explanation will receive few, if any, marks. Short, elegant solutions will gain more marks than long, contorted ones.

You may use a calculator in this paper.

Questions

1. Calculate $1\frac{1}{2} \times 1\frac{1}{3} \times 1\frac{1}{4} \times 1\frac{1}{5} \times 1\frac{1}{6} \times \ldots \times 1\frac{1}{2011}$.

[3 Marks]

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2. Find a single digit number which, when raised to the fifth power, is equal to 32768.

[3 Marks]

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3. Which number gives the same result when it is added to $3\frac{3}{4}$ as when it is multiplied by $3\frac{3}{4}$?

[3 Marks]

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4. What is the sum of all the numbers from 1 to 2011?

[3 Marks]

5. If eight men can dig 16 holes in 32 days, how long will it take four men to dig eight holes of the same size?

[3 Marks]

6. Using the digits 1 to 9, and just one mathematical 'sign', make an expression exactly equal to 100.

[3 Marks]

7. If P, Q and R represent three different digits, and QR represents the number obtained by simply placing Q and R together (rather than multiplying them together), what are the values of P, Q and R given that $(QR)^2 = PQR$?

[3 Marks]

8. The colours used in traffic lights are red, amber and green, and are used singly or in combination – four in practice (red only, red and amber, green only and amber only). How many signals could be represented if all possible combinations were allowed, assuming that at least one colour shows?

[3 Marks]

9. If, in a race, the man who came two places in front of the last man finished one place ahead of the man who came fifth, how many men finished the race?

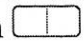
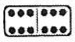
[3 Marks]

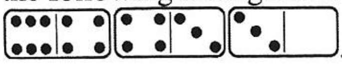
10. A man ate 100 grapes in five days, each day eating 6 more than on the previous day. How many grapes did he eat on the first day?

[4 Marks]

11. If a rectangular carpet has an area of $120m^2$ and a diagonal measuring exactly 17m, what are the carpet's dimensions?

[4 Marks]

12. A set of dominoes consists of 28 rectangular tiles. Each tile has two squares, each with a whole number of dots between 0 and 6. Every combination of two numbers from  (0-0) to  (6-6) is represented.

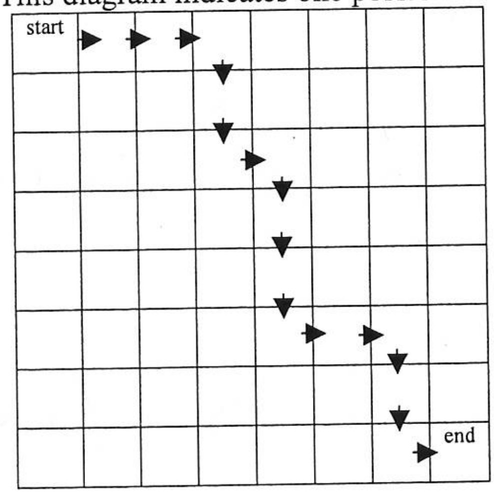
Dominoes may be placed alongside each other only if the numbers on the touching squares match. For example, the following arrangement is allowed because the "4"s match, and the "3"s match: .

If you place all 28 dominoes in a line (remembering that touching squares must match) so that there are 5 dots at one end, how many dots must be at the other end of the line?

[5 Marks]

13. How many different ways are there of moving from the top left square of a chessboard to the bottom right square, if moves are made only one square at a time and either right or down? (ie each journey covers 14 squares).

This diagram indicates one possible route.

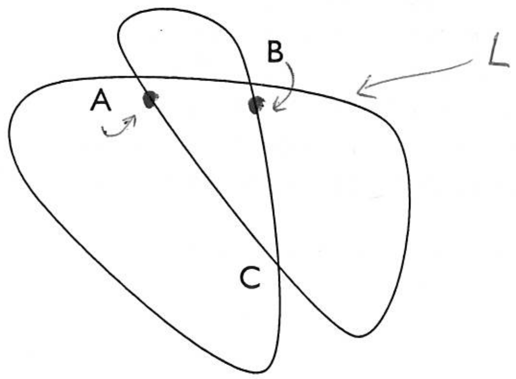


[5 Marks]

14. Find the two smallest whole numbers such that the difference of their squares is a cube, and the difference of their cubes is a square.

[5 Marks]

15. A loop of string is lying on the ground in the manner shown. The loop is too far away for you to determine how it crosses itself at points A, B and C. What is the probability that the string is knotted?



[5 Marks]

16. There are only two right-angled triangles whose sides are integers and whose area is the same as its perimeter. One of them is the 5:12:13 triangle, whose area = perimeter = 30. What are the lengths of the sides of the other triangle for which this is true?

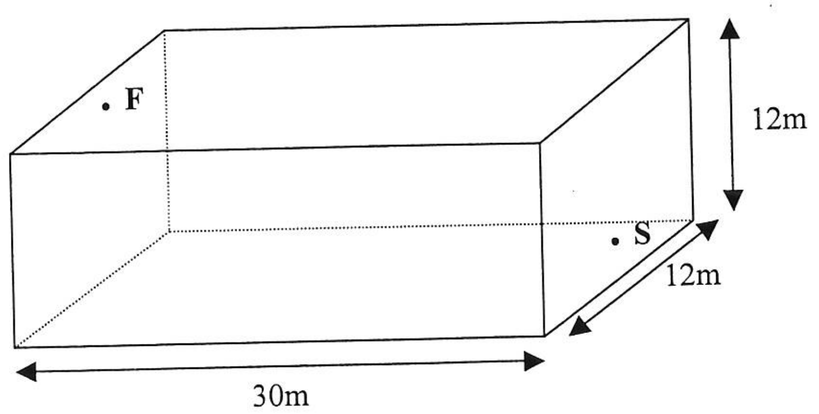
[5 Marks]

17. The Mathschester River has two bridges across it that are exactly 1 km apart. One day, while practising for an important sculling competition, Adam Buster rowed upstream. He always rowed at a constant rate, and in doing so passed under the two bridges on the way upstream. Just under the second (upstream) bridge his rubber duck (a good luck charm) fell into the water. A further ten minutes passed before Adam realised that he had lost his rubber duck, whilst he had continued to row upstream. Adam then turned round (instantaneously) and immediately began rowing downstream, still at a constant rate, in the direction from which he had come. He finally caught up with the rubber duck under the first (downstream) bridge.

How fast does the Mathschester River flow?

[5 Marks]

18. Inside a room 30m long, 12m wide and 12m high, a spider (S) catches sight of his supper, a fly (F). The spider is located on of the end walls, 1m from the floor, midway between the side walls. The fly, paralysed with fear, is on the opposite end wall, midway between the two side walls, and 1m from the ceiling.
What is the shortest distance that the spider must crawl to reach the fly?

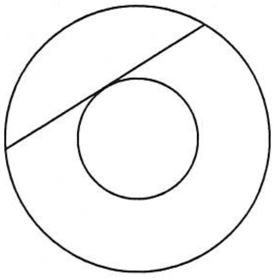


[6 Marks]

19. After throwing a dart n times at a dartboard, my percentage of hits is $p\%$. With my next throw I make a hit and my percentage of hits is now $(p+1)\%$. What is the value of $n+p$?

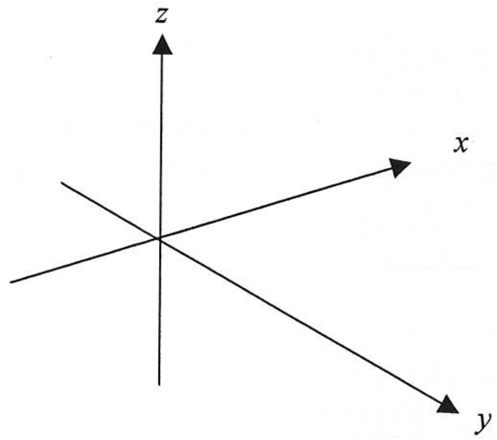
[8 Marks]

20. Two concentric circles have radii of 10cm and 26cm. What is the length of the longest straight line that can be drawn between them?



[8 Marks]

21. In a three-dimensional co-ordinate system the point P has co-ordinates (x, y, z) where x, y and z are three consecutive positive integers.
If the distance of the point P from the origin is at least 175, find the minimum possible value of x.



[8 Marks]

22. Some sequences converge to a single limiting value, as $n \to \infty$. For example, $a_n = \frac{2^n}{3^n}$ will converge to **0**, because the sequence can be rewritten as $a_n = \left(\frac{2}{3}\right)^n$.

Another example is $a_n = \frac{12(n!)}{n^5 + 3(n!)}$, which converges to **4**.

This is because $n!$ (ie $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$) is significantly more dominant than n^5, so we can 'ignore' the effect of n^5 on the sequence and just consider $n!$ terms, as follows:

$$a_n = \frac{12(n!)}{n^5 + 3(n!)} \approx \frac{12(n!)}{3(n!)} \Rightarrow \frac{12}{3}\frac{n!}{n!} \Rightarrow 4$$

The following sequence also converges to a single value:

$$a_n = \frac{n^4 - n^2 + 6(n!)}{2n^3 - 2(n!) + 7n}$$

Using some of the ideas shown above, what is the convergence value?

[5 Marks]

END OF THE PAPER