Arnold Hagger Mathematics Prize

27th January 2010

90 minutes

- All solutions must be written in the answer booklet provided.
- Answer as many questions as you can.
- Answers without explanation will receive no marks.
- Full marks will be awarded only for clear, logical and elegant solutions.
- It is better to produce a few complete solutions than a large number of incomplete solutions.
- There are 150 marks available.
- You are not necessarily expected to finish the paper.
- Calculators and standard geometrical instruments are allowed.

1) The numbers 2, 3, 5, 5 and x have the same mean, median and mode. What is x?

[5 marks]

2) What fraction is halfway between $\frac{1}{3}$ and $\frac{2}{5}$?

[5 marks]

3) Last year I went on holiday to the Lake District and suffered from the usual indifferent weather. (Counting only those days that I was there for the full 24 hours) I noted that it rained at some time on 9 of the days, that there were 10 clear mornings and 9 clear afternoons. Fortunately, when it rained in the morning it was always clear in the afternoon. For how many days was I on holiday?

[5 marks]

4) At a party each guest shakes the hand of all the other guests that he/she meets (though guests do not necessarily meet all the other guests). Explain why the number of guests who shake an odd number of hands must be even.

[5 marks]

5) I have a square piece of metal. From it I cut out a circle of the maximum possible size. From this I cut out a square of the maximum possible size. Then I cut out another circle, etc. How many circles have I cut out when the remaining square has sides $\frac{1}{8}$ of the length of those of the original square?

[5 marks]

6) Solve $\frac{1}{(x+1)^2}+\frac{1}{(x-1)^2}=1$, giving the answers exactly.

[10 marks]

7) A 6-digit number is divisible by 37. It is cut in half to make two 3-digit numbers. Prove that the sum of these two 3-digit numbers is also divisible by 37.

[10 marks]

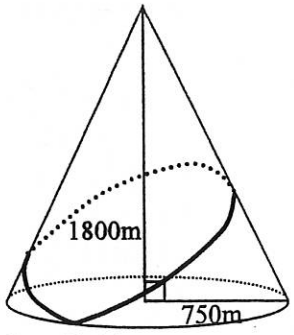
8) Without using a calculator, work out which is larger of the two numbers 2^{85} and 3^{53}.

[10 marks]

9) Show that the line $(k-2)y+kx=k+6$ passes through one point for all values of k and find the coordinates of that point.

[10 marks]

10) An explorer stands at the base of a right conical hill that is 1800m high and has base radius 750m. He walks up and around the hill and back to his starting point. What is the length of the shortest route he can take?



[10 marks]

11) Show that every right-angled triangle whose sides are integers has at least one side whose length is a multiple of 5.

[15 marks]

12) You are given that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\frac{1}{6}\pi^2$$

where the left-hand side is the infinite series continuing in the obvious pattern. What is the value of the infinite series

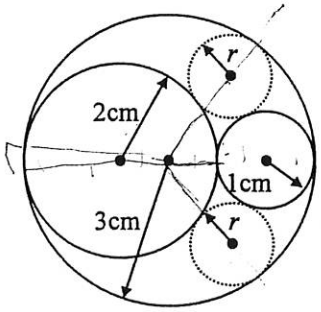
$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots?$$

[15 marks]

13) If $bc=a^p$, $ca=b^q$ and $ab=c^r$ (with $a,b,c>0$ & $a,b,c\neq 1$) show that $pqr=p+q+r+2$.

[20 marks]

14) Inside a hollow cylinder of radius 3cm are placed two cylinders with radii 2cm and 1cm. What is the radius r of the largest cylinder that would fit in the remaining two gaps?



[25 marks]