Arnold Hagger Mathematics Prize

24th January 2007

90 minutes

- Answer as many questions as you can.
- All answers must be written in the answer booklet provided.
- There are 100 marks altogether.
- You are not necessarily expected to finish the paper. Producing a few complete, elegant solutions is better than doing scraps from each question.
- A calculator may be used in any question, but will not be very useful (!)
- Standard geometrical instruments are also allowed.

Good luck! **JCA**

1) Order these numbers from smallest to biggest, showing your working:

$$2^{5000} \qquad 5^{2000} \qquad 3^{3000}$$

[5 marks]

2) If the number $2^{1005} \times 5^{995}$ is written out in full, how many digits will it have?

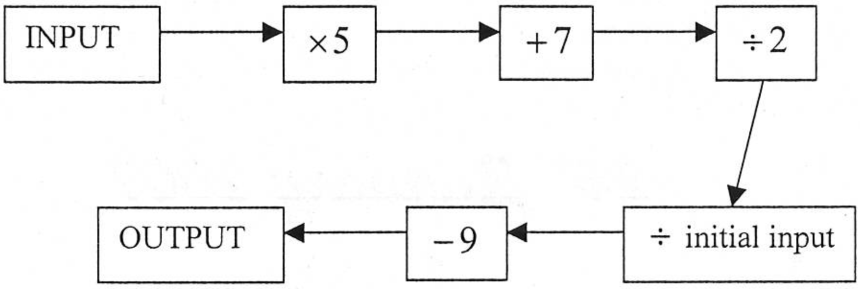
[5 marks]

3) a) Multiply out $(x+3)(x^2+x+3)$ and simplify your answer.

b) Using your answer to part (a), explain carefully whether or not 1469 is a prime number.

[5 marks]

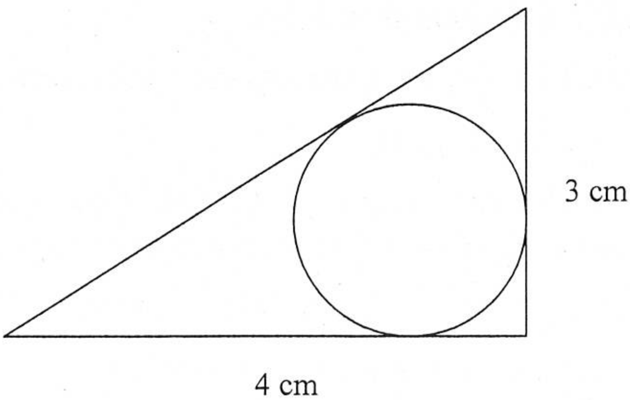
4) A simple machine performs the following calculations:



If the output is equal to the input, what can you say about the input?

[5 marks]

5) Calculate the exact area of the circle in the triangle below:



[5 marks]

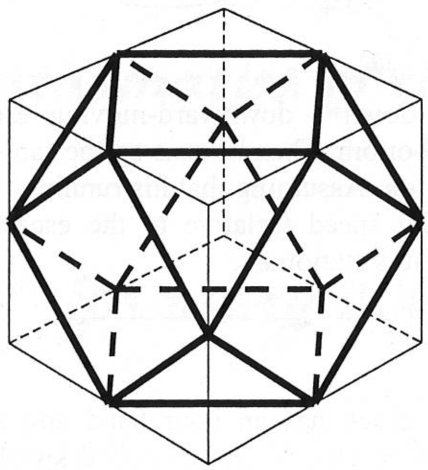
6) One of the pieces in a jigsaw is in the shape of a regular n-sided polygon. If this piece is removed and rotated by 75° about its centre, it will fit back exactly in its original place in the jigsaw. What is the smallest possible value of n?

[5 marks]

7) Dynamite used in a quarry is supplied with a fuse which burns at a non-uniform rate, and takes 4 minutes to burn fully along its length. Given two such fuses (without the dynamite!) and a box of matches, how can you measure a time period of 1 minute?

[5 marks]

8) The bold lines in the diagram show a regular **cuboctahedron**. This solid has 14 faces (6 square and 8 triangular) and is formed by joining up the midpoints of all pairs of adjacent edges of a cube.



The outer cube has edges of length 2 units. Calculate exact expressions for:

a) the proportion of the cube occupied by the cuboctahedron;

b) the distance between any two opposite triangular faces of the cuboctahedron.

[10 marks]

9) On a "lucky dip" UK lottery ticket, 6 different whole numbers are selected at random between 1 and 49 inclusive. Find the probability that:

a) **no** two of the 6 numbers are consecutive;

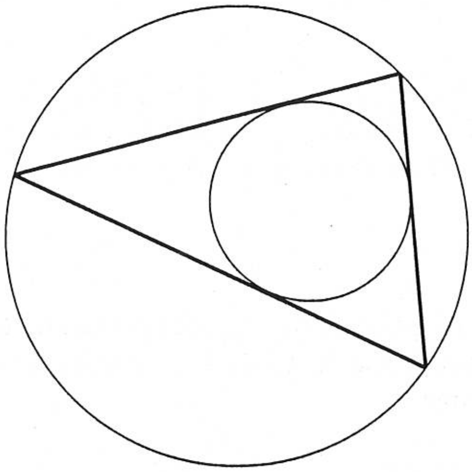
b) there is a sequence of four (but not five) consecutive numbers.

Give exact, simplified expressions.

[10 marks]

Please turn over.

10) Triangle ABC has sides of length a, b, c. The circumcircle of the triangle has radius R and the incircle has radius r. The semi-perimeter s of the triangle is defined to be $s = \frac{1}{2}(a+b+c)$. Prove that $abc = 4Rrs$.



[10 marks]

11) DangerMouse walks down a downward-moving escalator, taking 14 steps of the escalator to reach the bottom. Then he runs up the same escalator, taking 84 steps of the escalator to reach the top. Assuming that his running speed (relative to the escalator) is four times his walking speed (relative to the escalator), find how many steps the escalator shows when it is stationary.

[10 marks]

12) An accurate 12 hour clock has an hour hand and a minute hand which both turn continuously at steady speeds. A time is called **special** if the hands can be swapped with each other to give a valid time.

Special times are quite rare. For example, 3:30 is **not** a special time. At 3:30, the hour hand points halfway between 3 and 4, and the minute hand points at 6. Swapping the hands here would result in an impossible time: if the hour hand was pointing at 6, the minute hand could not point halfway between 3 and 4 because it would have to point at 12.

During a whole day, on how many occasions will the clock show a special time?

[10 marks]

13) For each three-digit positive integer N, define $f(N)$ to be the sum of the *hundreds* digit, the square of the *tens* digit, and the cube of the *units* digit, of N.

For example, $f(312) = 3 + 1^2 + 2^3 = 18$. 12

Determine, with proof, all such N for which $f(N) = N$.

[15 marks]

++++ **The End** ++++