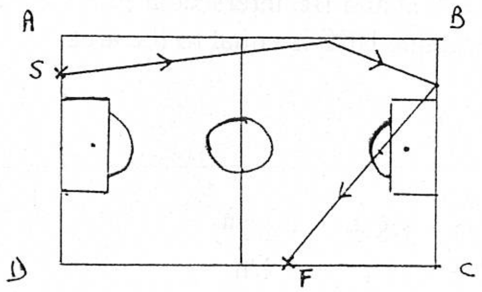
SHREWSBURY SCHOOL
MATHEMATICS DEPARTMENT

ARNOLD HAGGER PRIZE
JANUARY 2000

Time 1 1/2 hours
Calculators are not to be used. Attempt as many questions as possible. Marks for each question or part question are shown in brackets. Your solutions must show full mathematical reasoning.

1. I look at my watch and observe that the hour and minute fingers are both pointing in the same direction. How long will it be until they first point in the opposite direction? [5]

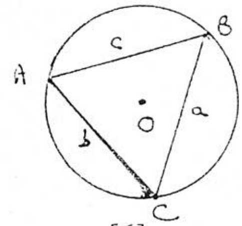
2. I am thinking of a three digit perfect square number. When it is rotated through 180^o it is also a perfect square. When the last two digits are swapped around it is also a perfect square. What is my number? [5]

3. 

ABCD is a rectangular soccer pitch. As part of a training exercise I have to sprint from S to F via a point on each of the sides AB and BC. I try and compensate for my lack of speed when compared to the rest of the group by running the least distance possible. Explain my strategy. [5]

4. Six guests (three husbands and their wives) sit down around a circular dining table. In how many ways can this be done?
In how many of these arrangements does each man sit next to his wife?
How many arrangements are there in which no man sits next to his wife? [9]

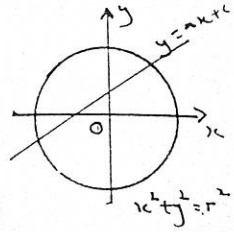
5. A,B and C are three points on the circumference of a circle centre O. With the usual notation AB=c, AC=b and BC = a.
The angles and sides are connected by the well known Sine Rule
a/sinA = b/sinB= c/sinC.
Prove that each of these ratios is equal to the diameter of the circle. [6]



6. x and y are two numbers with the property that $x + y = 3$ and $xy = 5$.
Find the values of
(i) $1/x + 1/y$, (ii) x^3+y^3. [6]

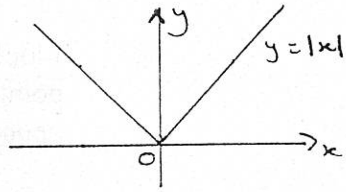
7. x is an integer between 10000 and 99999. If x is written as abcde, where a,b,c,d,and e are integers between 0 and 9 ($a>0$) prove that if x is divisible by 11 then $a - b + c - d + e$ must be divisible by 11. [6]

8. x and y are two positive numbers. Prove that $(x + y)/2 > \sqrt{(xy)}$. [5]

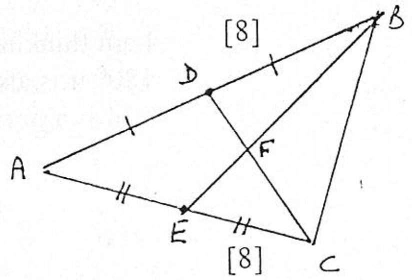


9. Find the values of c for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = r^2$. [6]

10. The modulus function $|x|$ is defined by the formula
$|x| = x$ if $x \geqslant 0$
$\quad -x$ if $x < 0$
and its graph is shown in the diagram opposite.
On separate diagrams sketch graphs of the functions
(i) $y = |1 - x|$ $(= | x - 1|)$, (ii) $y = 1 - |x|$
and (iii) $|y| + |x| = 1$. [8]



11. The sides AB,AC of a triangle are bisected at D and E respectively and the lines CD and BE intersect at F. Prove that the area of triangle BFC is equal to the area of quadrilateral ADFE. [8]



12. Consider the two series
(i) $S = 1 + 1/2 + 1/4 + 1/8 + ... + 1/2^n$,
(ii) $T = 1 + 1/2 + 1/3 + 1/4 + ... + 1/n$.
Investigate what happens to the value of S and T as the value of n increases.
Prove any assertion that you make. [8]

13. The sequence of numbers $x_1, x_2, x_3, \ldots$ is generated by the formula
$x_{n+2} = x_{n+1} + x_n$ (1)
where $x_1 = x_2 = 1$.
Find the two values of α (call them α_1 and α_2) such that $x_n = \alpha^n$ is a solution of equation (1).
Assuming that the general solution of equation (1) may be found in the form $x_n = A\alpha_1{}^n + B\alpha_2{}^n$, where A and B are constants, use the given values of the first two terms x_1 and x_2 to determine the values of A and B. [8]