Shrewsbury School Presents

The Arnold Hagger

Mathematics Prize

Wednesday 23rd February 1994

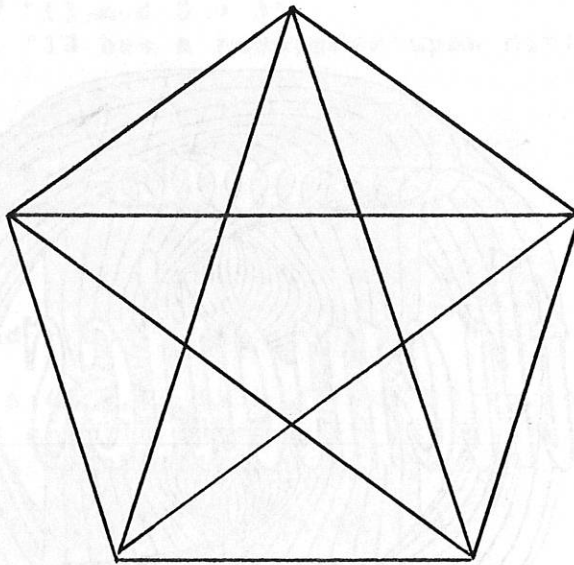## 7.15pm - 8.45pm

M4

Calculators may NOT be used

* You have 1 hour and 30 minutes to attempt as many questions as you like.

* Questions may be answered in any order.

* Make your methods of solution clear by including all working and reasoning.

* Calculators may NOT be used.

Question one : Count Triangular.
===(6 marks)

How many different triangles are there in the following figure ?

Question two : 101.
==================(6 marks)

How many positive integers less than 101 are not divisible by 3 or by 5 ?

Question three : Give and Take.
==================================(6 marks)

Here is a curious nonsense; 1 2 3 4 5 6 7 8 9 = 100.

There is only one way to insert a total of 7 plus and minus signs between the digits on the left side to make the equation correct.

It is ;   1 + 2 + 34 - 4 + 5 + 6 + 78 + 9 = 100.

There is only one way to insert a total of 3 plus and minus signs between the digits on the left side to make the equation correct.

What is it ?

Question four : Simply Simultaneous.

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Here are two equations which are to be solved simultaneously;

$$6751x + 3249y = 26751$$
$$3249x + 6751y = 23249$$

Is this a joke ?

Not if you can multiply the first equation by 6751 and the second by 3249, and not if you use a second, simpler method.

Question five : A Guessing Goose.

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A lone goose met a flock of geese flying in the opposite direction.

"Hello 100 geese !", he cried.

The leader of the flock replied, "We are not 100. If you take twice our number and add half our number and add a quarter of our number and finally add you, then we are 100".

How many geese are in the flock ?

Question six : Water on the Brain.

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A man goes to a stream with a 9 litre container and a 16 litre container.

What should he do to get precisely 1 litre of water in the 16 litre container ?

Question seven : Four in One.

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Four circles of radius 1m just fit together inside a circle of radius r as shown in the diagram below.

Calculate r.
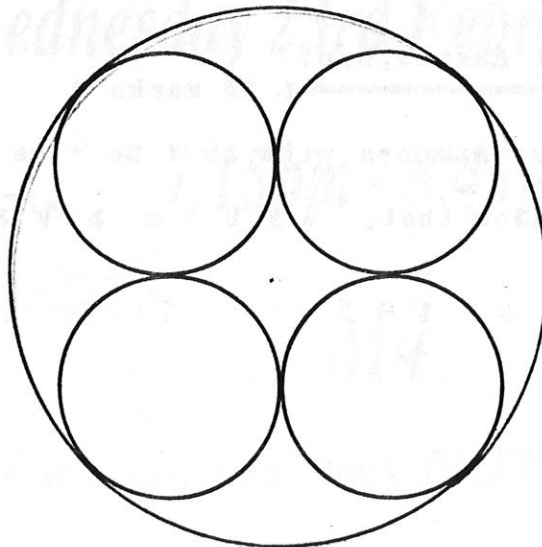
Question eight : Just Before Midnight.
(10 marks)

A clock rests on a stand so that its face makes an angle of 45 degrees to the horizontal.

Find the angle made with the horizontal by the minute hand seven and a half minutes before midnight.

Question nine : Prime Result.
(10 marks)

Prove that $P \mod 2P^2 = P$ where P is a prime number.

Note : The "mod" function means "has a remainder upon division by".
For example, "13 mod 5 = 3"
means "13 has a remainder upon division by 5 of 3"

Question ten : Triple Trouble.
(15 marks)

Show that
$$(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$$

A Pythagorean Triple (x,y,z) is a set of three positive integers x,y and z such that x,y and z are the sides of a right angled triangle. (We will assume that z is the hypotenuse).

Two such triples are (3,4,5) and (5,12,13).

From the first part of this question I can deduce that (16,63,65) is another Pythagorean Triple.

How are 16, 63 and 65 related to the numbers of the two original triples (3,4,5) and (5,12,13) ?

Find a Pythagorean Triple whose hypotenuse (the z value) is 845.

Question eleven : A Not Easy a,b,c.
(15 marks)

a, b, and c are positive numbers with ab + bc + ca = 1.

Show that; $a + b + c \geqslant \sqrt{3}.$