SHREWSBURY SCHOOL.

THE ARNOLD HAGGER MATHEMATICS PRIZE EXAMINATION.

Wednesday 5th.March 1986.

Time allowed : $1\frac{1}{2}$ hours.

Answer as many questions as you like, in any order. All answers must be fully explained. The number of marks available fot each question is shown in brackets.

CALCULATORS MAY BE USED.

You may find the following information helpful :

An INTEGER is a member of the set $\{\ldots,-3,-2,-1,0,1,\ldots\}$

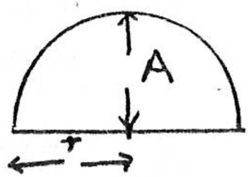
An IRRATIONAL number is a number which cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.

Area of a triangle = 1/2 base x height.

Volume of a cylinder = $\pi r^2 h$.

Volume of a sphere = $4/3 \times \pi r^3$.

Volume of a 'cap' of a sphere, as shown = $\pi A(3r^2 + A^2)/6$.



A straight line, drawn parallel to one side of a triangle, divides the other sides proportionally.

TURN OVER, WHEN YOU ARE TOLD TO DO SO.

1. Which years in this decade i.e. 1980-1989, can be expressed as the sum of 2 squares? Where this occurs give the squares involved. (Note: No integer which can be written in the form $4k+3$ can be expressed as the sum of 2 squares.)

(10 marks)

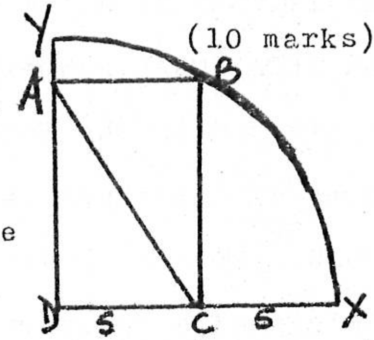
2. Four prize winners are to receive their prizes from a V.I.P., who does not know them. If he gives out the prizes at random, what is the probability that everyone gets the wrong prize?

(10 marks)

3. Prove that 3^n has a remainder of 1 when divided by 8 if n is even; and a remainder of 3 when divided by 8 if n is odd.

(10 marks)

4. Rectangle ABCD is inscribed in the quadrant YBX of a circle as shown. Given the distances as indicated, what is the area of triangle ACD? (give your answer correct to 3 sig.figs.)



(10 marks)

5. Prove algebraically that any 5 digit number made up of 5 identical digits cannot be divided exactly by the sum of its digits. e.g. $44444 \div (4+4+4+4+4)$ does not have an integer solution.

(10 marks)

TURN OVER

6. What is the smallest number which, when divided by 10 gives a remainder of 9; when divided by 9 gives a remainder of 8; when divided by 8 gives a remainder of 7 and so on, down to a remainder of 1 when divided by 2 ?

(10 marks)

7. Prove carefully that $\sqrt{2}$ is irrational.

(10 marks)

8. ABCD is a quadrilateral with $\angle ADC = \angle ABC = 90^\circ$, and each side measures an integer number of metres. What is the least possible area of ABCD ?

(10 marks)

9. Show that, if n is a non-zero integer, then

$$x^2 - y^2 = n$$

is soluble (in integers x, y) if and only if n is odd or 4 divides n exactly.

(10 marks)

10. A.B = B; B.C = AC; C.D = BC; D.E = CH; E.F = DK; F.H = CJ; H.J = KJ; J.K = E; K.L = L; A.L = L; Every letter represents a different digit; A.B means A multiplied by B etc. AC, BC etc. represent 2 figure numbers. Find A, B, C, D, E, F, H, J, K, L.

(15 marks)

11. A cylindrical hole, 6 cms. long, has been drilled straight through the centre of a solid sphere. What is the volume remaining in the sphere ?

(15 marks)

12. Find <u>all</u> the integer solutions of:

$$x^2 - xy - 12y^2 = 527.$$

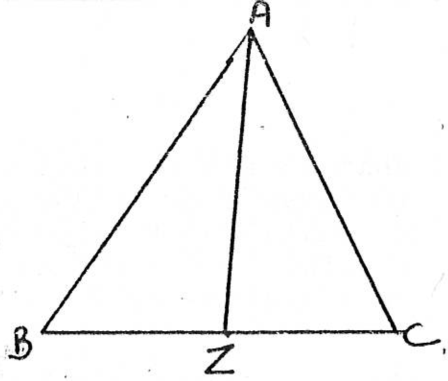
(15 marks)

TURN OVER

13. AZ is the bisector of ∠BAC as indicated. Show that AZ divides BC in the same ratio as the sides containing ∠BAC. i.e. show that

$$\frac{BZ}{ZC} = \frac{BA}{AC}$$

(hint: construct a line through C, parallel to ZA.)



(15 marks)

14. Three men, Arthur, Bernard and Charles, with their wives, Anne, Barbara and Cynthia (not necessarily respectively) make some purchases. When their shopping is finished, each finds that the average cost in pounds of the articles he or she purchased is equal to the number of his or her purchases. Arthur has bought 23 more articles than Barbara and Bernard has bought 11 more articles than Anne. Each husband has spent £63 more than his wife. Who is married to whom ?

(15 marks)
