SHREWSBURY SCHOOL 1984 MATHEMATICS PRIZE

Thursday 2nd. February $1\frac{1}{2}$ hours

You may attempt as many questions as you wish, in any order. Complete answers to a few questions will receive more credit than a large number of fragmentary solutions. Be careful to make your methods clear by including all your working and reasoning ; answers alone will receive little credit.

CALCULATORS AND TABLES MAY NOT BE USED.

You may find the following information helpful :

A NATURAL number is a member of the set $\{1,2,3,4,.....\}$

An INTEGER is a member of the set $\{.....,-3,-2,-1,0,1,2,....\}$

A RATIONAL number is a number which can be expressed in the form $\frac{a}{b}$, where a and b are integers.

An IRRATIONAL number is a number which cannot be expressed in the form above.

A sequence of numbers is in ARITHMETIC PROGRESSION if the difference between any two successive numbers is the same throughout the sequence.

If A and B are two points on the circumference of a circle, C is another point on the circle, on the longer arc from A to B, and O is the centre of the circle, then $\hat{AOB} = 2\hat{ACB}$.

A PRIME number is a natural number divisible only by itself and 1. Note that 1 is not a prime number.

TURN OVER , when you are told to do so by the invigilator.

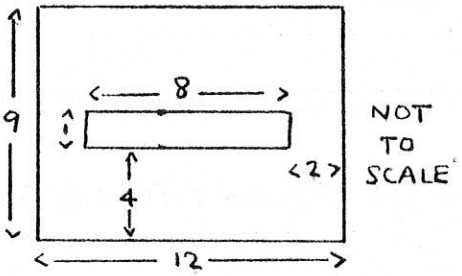
There are TEN questions on the paper.

1. (a) Find 3 integers in arithmetic progression whose product is a prime number.

(b) Prove that it is impossible to find 4 integers in arithmetic progression whose product is a prime.

2. Divide the shape shown into 2 regions which will fit together to form a square measuring 10x10 units^2.

You will need to make a copy of the diagram.



3. At a dinner party, 4 couples are present. They sit at a circular table set for 8. One place is reserved for the host. In how many different ways can the remaining 7 people be seated if

(a) No 2 men sit next to each other ?

(b) No 2 men nor any married couple sit next to each other ?

4. Find the smallest natural number such that

(a) When divided by each of the numbers 2,3, and 4 it leaves remainder 1, but is divisible by 5.

(b) When divided by each of the numbers 2,3,4,5, and 6 it leaves remainder 1, but is divisible by 7.

5. (a) The number x satisfies $x^2 - 2x + 7 = 0$. Express x^6 in the form $ax + b$.

(b) The roots of $x^2 - 2x + 7 = 0$ are α and β. What is the quadratic equation with roots $1/\alpha$ and $1/\beta$?

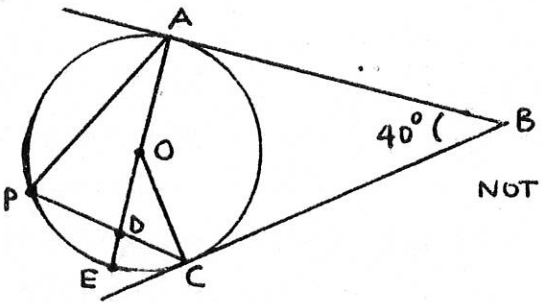
6. Y is an irrational number.

(a) Prove that, given any integer a, there exists an integer b such that $0 < aY + b < 1$.

(b) Suppose that we have found integers a, b, c and d such that $0 < aY + b < 1$; $0 < cY + d < 1$, $a \neq c$

Prove that $aY + b \neq cY + d$.

7. In the configuration shown, O is the centre of the circle, and the triangle ADP is isosceles. Find the angle $\hat{PAB}$.



8. The number ****326743 (where each * represents a digit) is a power of a prime number less than 12. The sum of the first 4 digits is 24.

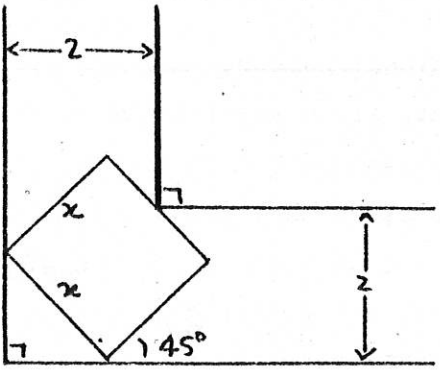
(a) What is the prime ?

(b) What is the power ?

9. Find all the solution pairs to the simultaneous equations $x^2 + y^2 = 5$; $xy + x + y = 5$, by using the substitutions $u = x + y$, $v = xy$.

TURN OVER

10. The figure ABCD is a square of side x. Find x.



NOT TO SCALE

JDS 1984