SHREWSBURY SCHOOL MATHEMATICS PRIZE 1980

A ttempt as many questions as you are able in the time allowed of one and one half hours. The questions are arranged roughly in ascending order of difficulty.

1) How many numbers between 1000 and 9999 inclusive contain two or more duplicated digits (i.e. digits occuring more than once)?

2) In a survey of 1000 mathematics masters, each was found to own at least one pullover from the following three types: Yellow, red and blue crested.

400 did not own a yellow pullover, 380 did not own a red and 542 did not own a blue crested pullover. 294 had both a red and a blue crested pullover, 277 had both a yellow and a red pullover and 190 had both a yellow and a blue crested pullover. How many had all 3? How many had only a red?

3) A prime number is one which is divisible by 1 and itself only. Show that if x and y are integers (whole numbers) which are odd then $x^2 + y^2$ is not prime. Show that it is not divisibly by 4.

4) The Fibonacci numbers are the sequence 1, 2, 3, 5, 8, 13, ... in which each integer (after the second) is the sum of the preceding two. Show that each consecutive pair of numbers in the sequence are coprime (have highest common factor one).

5) If $\alpha, \beta$ are the roots of

$$x^2 - 2x - 1 = 0$$

and $s_n = \alpha^n + \beta^n$

prove that $s_n - 2s_{n-1} - s_{n-2} = 0$

Hence evaluate $\alpha^3 + \beta^3$ and $\alpha^5 + \beta^5$ .

6) If m is a positive integer prove that

$$7^m(3m + 1) - 1$$

is always divisible by 9.

7) Show that if n is integral (a whole number) then $\sqrt{n}$ is either irrational or integral. (irrational means cannot be expressed as a ratio of two integers )

8) The rational numbers x, y (rational means not irrational) satisfy the equation

$$x^2 - 2xy - 2y^2 = 0$$

prove that $x = y = 0$ .

9) When n = abcdef ( in base 10), F(n) = abc - def. For example if n = 456123 then F(n) = 456 - 123 = 333.

Show that n divides exactly by 7 if and only if F(n) divides by 7. What other numbers could replace 7?

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10) Show that if $2^n - 1$ is prime then n is prime. (A prime number was defined in question 3 ) Is the converse true?

An integer is called perfect if the sum of all its proper divisors (i.e. all its divisors excluding itself) add up to the original number.
For example 6 (= 1 + 2 + 3 )
and 28 (=1 + 2 + 4 + 7 + 14) are perfect numbers.
Show that if $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is perfect .

11) Show that if I place n goldfish in n-1 goldfish bowls then at least one goldfish bowl will contain ~~two~~ $>1$ fish.

An equilateral triangle of side one unit contains 5 points randomly distributed throughout the triangle. Show that there are two points which are less than (or equal to) $\frac{1}{2}$ unit apart.

GOOD LUCK !

M.T.