F.M.H. Note misprint on Qu 5(i). I have corrected all copies. RH.

SHREWSBURY SCHOOL MATHEMATICS PRIZE 1978.

Thursday, February 23rd.. Time allowed: 90 minutes.

ATTEMPT AS MANY QUESTIONS AS YOU LIKE. THE QUESTIONS ARE ARRANGED IN ASCENDING ORDER OF DIFFICULTY, AND DO NOT NECESSARILY CARRY EQUAL MARKS. SHOW ALL WORKING, AND EXPLAIN WHAT YOU ARE DOING. ANSWERS TO FULL QUESTIONS WILL RECEIVE FAR GREATER CREDIT THAN FRAGMENTARY SOLUTIONS.

1. (i) Prove that if a number is divisible by 3, then the sum of its digits must also be divisible by 3.

   (ii) Prove that, for any positive integer n, the number $n^2(n+1)^2(2n^2+2n-1)$ is divisible by 12.

2. (i) Prove that 121 represents a perfect square in every number base greater than 2.

   (ii) Let AB be any number between 10 and 99 inclusive. Prove that the number ABABAB is divisible by 7. (For example 484848 = 69264 x 7)

3. (i) Three people in the N*W H**SE purchase a barrel of ale. One of them, the setter of this problem, is an advocate of total abstinence from intoxicants, but the other two are not. The barrel contains 8 gallons of beer, and the men wish to divide the ale into two equal amounts without wasting a drop of it. They are able to make use only of the barrel itself and two empty jugs, of capacities 5 gallons and 3 gallons. Show them how to do this.

   (ii) Aelfric, Beowulf, Guthlac, and Hrothgar each keep a cat or a dog (but not both). They all live in the village of Swarthbeck, where cat owners always tell the truth but dog owners never do. I asked each of them recently whether the other three all keep a cat. Each replied: "No." How many keep cats ?

4. A ring has 132 gear teeth on its inside surface. The teeth are equally-spaced, and the ring is fixed onto a piece of paper. A cog wheel has 24 compatible teeth evenly spaced out on its perimeter. This wheel is rotating, without slipping, so that its perimeter is always in contact with the inside surface of the ring. What path is traced out by a point on the circumference of the wheel? Draw a sketch of this curve, and say how many cusps (i.e. outer points) it has. Which other cog wheels would produce curves with the same number of cusps when rotated inside this ring?

5. (i) A prime number is a number which is divisible only by itself and by the number 1. It is easy enough to find three consecutive integers, none of which is prime, eg. 20, 21, 22. Prove that it is possible to find a set of one million consecutive integers, none of which is prime.

(ii) What is the exact time between 5 o'clock and 6 o'clock when the two hands of the School clock are in the same position?

6. Consider the number 3529411764705882. When its first digit is moved into the last place, we obtain the number 5294117647058823, which is exactly one and a half times the first number. Find a number N which has the property that the new number formed from N, when N's first digit is moved into the last position, is exactly three times the number N.

TURN OVER.

7. A disc may be divided into at most two parts by a single cut, and into at most four parts by two cuts. What is the greatest number of parts into which a disc may be divided by: (a) 3 cuts, (b) 1000 cuts, (c) 10,000 cuts ?
Find a formula p(n) for the maximum number of parts into which a disc may be divided by n cuts.

8. Six distinct points are chosen at random on a plane. Prove that it is possible to find a straight line which partitions the plane in such a way that 3 of the points lie on one side of the line, and the other 3 points lie on the other side of the line.
Prove also that if 1,000,000 distinct points are chosen at random on a plane, then we can find a straight line which partitions the plane in such a way that 500,000 of the points lie on one side of the line, and the other 500,000 points lie on the other side.

9. ABC is a plane triangle. D is a point on AB such that AD = 3DB ,
E is a point on BC such that BE = 3EC ,
F is a point on CA such that CF = 3FA ,
G is a point on DF such that DG = 3GF ,
H is a point on ED such that EH = 3HD ,
I is a point on FE such that FI = 3IE .
Prove that triangle GHI is similar to triangle ABC , and find the ratio of their areas.

R.H.