MATHEMATICS PRIZE 1975 Thu Feb 27th $1\frac{1}{2}$ hrs FMH

Attempt as many questions as you can. They do not necessarily carry the same number of marks.

1. Consider the sequence of fractions

$$\frac{2}{1},\quad \frac{2}{1+\frac{2}{1}},\quad \frac{2}{1+\frac{2}{1+\frac{2}{1}}},\quad \frac{2}{1+\frac{2}{1+\frac{2}{1+\frac{2}{1}}}},\ \text{etc.}$$

and so on. Let the n th fraction in the sequence be called u_n. Assuming that as n gets larger and larger, u_n approaches a value u, find the value of u and prove that it has this value. (Little credit will be given for working out the first few u_n and guessing.)

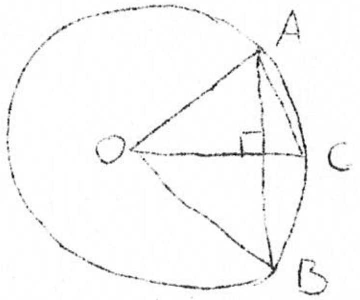
2. We have $n+1$ points $P_0, P_1, \ldots, P_n$ on a line. Each point is numbered 0 or 1, arbitrarily. Show that if P_0 is numbered 0 and P_n is numbered 1 then however we number $P_1, P_2, \ldots, P_{n-1}$ there are always an odd number of the segments P_0P_1, $P_1P_2, \ldots, P_{n-1}P_n$ which have a 0 at one end and a 1 at the other.

3. When written in base 9, prove that a perfect square can only end in certain digits, and show which these digits are.

4. Prove that $x = p + q$ is a solution of $x^3 - 3pqx - (p^3 + q^3) = 0$. Find p^3 and q^3 if ~~there~~ this is $x^3 - 6x - 6 = 0$ and hence find the real solution to the equation $x^3 - 6x - 6 = 0$.

5. The Fibonacci sequence $u_1, u_2, u_3, \ldots$ is defined by $u_0 = u_1 = 1$, $u_i = u_{i-1} + u_{i-2}$
If $u_i = \lambda^i$ find the 2 possible values λ_1 and λ_2 for λ.
Prove that u_i must be of the form $A\lambda_1^i + B\lambda_2^i$ and hence find u_i explicitly (finding A and B).

6. The circle has unit radius and $A\hat{O}C = \frac{1}{2}A\hat{O}B$, $AB = x$ and $AC = y$.
Prove that
$$y^2 = 2 - \sqrt{4 - x^2}.$$
Explain carefully how this can be used to give an approximate value for π.



7. If I have 6 beads of different colours in how many ways can I arrange them on a circular ring? In how many of these ways are the blue and the red beads together?

8. If the area of any triangle is called A and half its perimeter is s, prove (i) the radius of the in-circle is A/s.

(ii) the radius of the circumcircle is abc/4A where a, b and c are the sides.

9. Prove Pythagoras' Theorem <u>carefully</u> from the figure to the right. (This is President Garfield's proof of the Theorem.)