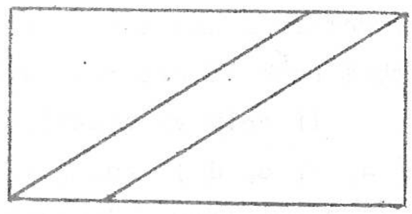
SHREWSBURY SCHOOL

MATHEMATICS PRIZE 1973

1. In a race (won by one horse only) a bookie quotes odds of 2 to 1 against horse A and 5 to 1 against B. What odds should he offer against A or B winning ?

2. A straight road is laid across a rectangular plot, 160 m by 50 m , in the manner shown in the diagram. Its area is found to be exactly one quarter of that of the original plot. What is its width ?



3. If $x^2 = 3x - 4$, prove (without solving the equation) that $x^3 = 5x - 12$. Explain why the converse is not necessarily true , and find the value of x which satisfies the second equation but not the first .

4. An insurance company classifies a set of 52 "lives" according as they are men (or women) , British (or foreign) , married (or single) and receives the information that there are:

Men 18 , Married 25 , British 41 , Married men 7 ,
British men 14, British married persons 16 , British married men 6 ,
Foreign single women 4 .

Show that these data are inconsistent . The company suspects a transcription error in the data of a single digit. Suggest a possible correction.

5. If four squares are placed externally on the four sides of any parallelogram , prove that their centres are the vertices of another square .

6. In the clan McBine , each man has two sons who have different names . The only Christian names given in the clan are Duncan , Ian and Keith . No son is named after his father . If S_p is the number of a clansman's (great $)^p$-grandsons who bear his Christian name, prove that

$$S_p = 2^p + S_{p-2} ,$$

and find the values of S_{2r} and S_{2r+1} .

TURN OVER

7. Given that

$$a + b + c + d = p + q + r + s \quad \dots\dots\dots\dots A$$

most of us have been tempted at some time to assume that

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + r^2 + s^2 \quad \dots\dots\dots\dots B$$

or even that

$$a^3 + b^3 + c^3 + d^3 = p^3 + q^3 + r^3 + s^3 \quad \dots\dots\dots\dots C$$

etcetera , and have been duly reproved . But it is possible to show that sets of natural numbers can be found , obeying such rules.

If only an equation of type A is true , we can call the sets (a, b, c, d) and (p, q, r, s) unigrade sets with 4 elements ; if A and B are true , duograde ; if A , B and C are true , trigrade , etc.

If (a, b) and (p, q) are unigrade sets, prove that (a, b, p+x, q+x) and (a+x, b+x, p, q) are duograde.

Show also that if (a, b, c) and (p, q, r) are duograde , then (a, b, c, p+y, q+y, r+y) and (a+y, b+y, c+y, p, q, r) are trigrade . The identity $(a + y)^3 = a^3 + 3a^2y + 3ay^2 + y^3$ can be quoted.

Starting with unigrade sets (1, 4) and (2, 3) and making x = 3 and y = 1 , obtain the trigrade sets (1, 5, 8, 4) and (2, 2, 7, 7) , and verify the results.